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PARTITION THEOREMS AND COMPUTABILITY THEORY

BY

JOSEPH ROY MILETI

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Abstract

The computability-theoretic and reverse mathematical aspects of various combinatorial principles, such as König's Lemma and Ramsey's Theorem, have received a great deal of attention and are active areas of research. We carry on this study of effective combinatorics by analyzing various partition theorems (such as Ramsey's Theorem) with the aim of understanding the complexity of solutions to computable instances in terms of the Turing degrees and the arithmetical hierarchy.

Our main focus is the study of the effective content of two partition theorems allowing infinitely many colors: the Canonical Ramsey Theorem of Erdös and Rado, and the Regressive Function Theorem of Kanamori and McAloon. Our results on the complexity of solutions rely heavily on a new, purely inductive, proof of the Canonical Ramsey Theorem. This study unearths some interesting relationships between these two partition theorems, Ramsey's Theorem, and Konig's Lemma, and these connections will be emphasized.

We also study Ramsey degrees, i.e. those Turing degrees which are able to compute homogeneous sets for every computable 2-coloring of pairs of natural numbers, in an attempt to further understand the effective content of Ramsey's Theorem for exponent 2. We establish some new results about these degrees, and obtain as a corollary the nonexistence of a "universal" computable 2-coloring of pairs of natural numbers.

To my parents

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Chapter 1

Introduction

1.1 Logic and Combinatorics

The connections between mathematical logic and combinatorics have a rich history. This dissertation focuses on one aspect of this relationship: understanding the strength, measured using the tools of computability theory and reverse mathematics, of various partition theorems. To set the stage, recall two of the most fundamental combinatorial principles, Ramsey's Theorem and König's Lemma. We denote the set of natural numbers by ω and use the notation $[A]^n$ to denote the set of subsets of A of size n.

- **Ramsey's Theorem** [23]: For every $n, p \geq 1$ and every $f\colon [\omega]^n \to p$, there exists an infinite $H \subseteq \omega$ such that f is constant on $[H]^n$. An H such that f is constant on $[H]^n$ is called *homogeneous* for f.
- **König's Lemma**: Every infinite bounded tree has a branch.

König's Lemma and Ramsey's Theorem are intimately related, as several proofs of partition theorems in set theory (such as Ramsey's Theorem) utilize paths through trees, and vice-versa. In the realm of large cardinals, those cardinals on which the appropriate analogue of Ramsey's Theorem hold are exactly those on which the appropriate analogue of König's Lemma hold (see [16, Theorem 7.8]).

Another interesting thread in the investigation of the logical strength of partition theorems is the use of infinitary methods to prove finite combinatorial results. One can use Ramsey's Theorem together with König's Lemma to derive the finite version of Ramsey's Theorem. This may seem like a curiosity since the finite version of Ramsey's Theorem can be proved using nothing but basic finite combinatorics, but the connection runs more deeply. In 1977, Paris and Harrington [21] proved that a natural finite partition theorem stronger than the finite version of Ramsey's Theorem is true but not provable in Peano Arithmetic, an important system which seems to correspond to our notion of finitary proof. Hence, any proof of this finite combinatorial result needs, in a precise sense, to use infinite sets. More recently, Kanamori and McAloon

[17] found another, perhaps more natural, finite partition theorem about regressive functions which is true but not provable in Peano Arithmetic.

Our interest is in the effective content of partition theorems such as Ramsey's Theorem. For example, we may ask whether every *computable* $f\colon [\omega]^2 \to 2$ must have a *computable* homogeneous set. If the answer is negative, we may wonder about the complexity of homogeneous sets for computable $f\colon [\omega]^2 \to 2$ as measured using the tools of computability theory, such as the Turing degrees and the arithmetic hierarchy. From a related but distinctly different perspective, we may seek to understand the strength of set existence axioms inherent in Ramsey's Theorem, as measured by tools of reverse mathematics. We might expect that the above mentioned relationship between partitions theorems and König's Lemma manifests itself in their corresponding computability-theoretic or reverse mathematical strengths.

In Chapter 2, we begin by discussing the various combinatorial principles that play a key role in this dissertation. We give proofs of König's Lemma and Ramsey's Theorem which will be relevant to later chapters, and summarize known results about the computability-theoretic and reverse mathematical strength of these principles. We also introduce the Canonical Ramsey Theorem and the Regressive Function Theorem.

In Chapter 3, we analyze the Canonical Ramsey Theorem, a generalization of Ramsey's Theorem due to Erdös and Rado in which the number of colors need not be finite. Known proofs of this result make use of Ramsey's Theorem (using Ramsey's Theorem for exponent $2n$ to obtain the Canonical Ramsey Theorem for exponent n), but we give a new purely inductive proof. Analyzing this proof, we provide upper bounds on the Turing degrees and position in the arithmetical hierarchy of canonical sets for computable partitions.

In Chapter 4, we analyze the Regressive Function Theorem of Kanamori and McAloon, whose finitary version was mentioned above. This result is an easy consequence of the Canonical Ramsey Theorem, so results on the Turing degrees and position in the arithmetical hierarchy of solutions for computable regressive functions give lower bounds for the Canonical Ramsey Theorem, aside from having intrinsic interest. We succeed in giving a sharp characterization of solutions for computable regressive functions for all exponents.

In Chapter 5, we study (s-)Ramsey degrees, i.e. those Turing degrees which are able to compute infinite homogeneous sets for computable (stable) $f\colon [\omega]^2 \to 2$. We show that these degrees are meager and have measure 0. We also improve various known results, and show that there is no "universal" computable partition, in contrast to the situation for Weak König's Lemma, the Canonical Ramsey Theorem, and the Regressive Function Theorem.

In Chapter 6, we study generalized notions of cohesiveness. Hummel and Jockusch studied higher analogues of cohesiveness for Ramsey's Theorem, and we initiate a study of analogues of cohesivness for the Canonical Ramsey Theorem and the Regressive Function Theorem.

In Chapter 7, we study the above partition theorems from the viewpoint of reverse mathematics. We show that many of the above principles are equivalent to ACA_0 over RCA_0, but a few special cases arise when considering exponent 1 and when quantifying over all exponents.

1.2 Notation and Conventions

We record here the notation and conventions that will be used throughout the dissertation.

Notation 1.2.1.

(1) ω denotes the set of natural numbers and $\mathcal{P}(\omega)$ denotes the set of subsets of ω. If we are working in a model $\mathfrak{M}$ of a subsystem of second-order arithmetic, we let $\mathbb{N}$ denote the set of natural numbers in the sense of $\mathfrak{M}$.

(2) We identify each $n \in \omega$ with its set of predecessors, so $n = \{0, 1, \ldots, n-1\}$.

(3) a,b,c,d,e,i,j,k,l,m,n,p,q,r,s,t,α,β,... will denote elements of ω (and sometimes -1).

(4) x,y,z,u,... will denote finite subsets of ω. We identify a finite subset of ω of size n with the n-tuple listing x in increasing order. When dealing with 1-element sets, we identify a and $\{a\}$.

(5) X,Y,Z,S,T,... will denote subsets of ω.

(6) A,B,C,H,I,J,M,V,... will denote infinite subsets of ω.

(7) Given $X \subseteq \omega$, let $X^{<\omega}$ be the set of finite strings of elements of X.

(8) $\sigma, \tau, \ldots$ will denote elements of $\omega^{<\omega}$. The empty string is denoted by ϵ. We identify $a \in \omega$ with the corresponding string in $\omega^{<\omega}$ of length 1.

(9) $\mathcal{D}$,$\mathcal{F}$,$\mathcal{M}$,$\mathcal{O}$,$\mathcal{U}$, ... will denote subsets of $\mathcal{P}(\omega)$.

(10) If x is a finite subset of ω of size n, and $i < n$, we denote the $(i+1)^{\text{st}}$ element of x, in increasing order, by $x(i)$.

(11) Suppose that $\sigma, \tau \in \omega^{<\omega}$. We write $\sigma \subseteq \tau$ to mean that τ extends σ, and we let $\sigma * \tau$ be the concatenation of σ and τ. We let $|\sigma|$ be the length of σ. If $i < |\sigma|$, we let $\sigma(i)$ be the $(i+1)^{\text{st}}$ element of σ.

(12) We identify each $X \subseteq \omega$ with its characteristic function, and hence we identify $\mathcal{P}(\omega)$ with 2^ω.

(13) If $\sigma \in 2^{<\omega}$ and $X \subseteq \omega$, we write $\sigma \subset X$ to mean that (the characteristic function of) X extends σ.

(14) If $\sigma \in 2^{<\omega}$, we let $\mathcal{I}(\sigma) = \{X \subseteq \omega : \sigma \subset X\}$ be the basic open neighborhood of 2^ω determined by σ.

(15) Suppose $X, Y \subseteq \omega$. Let $X \backslash Y = \{a \in \omega : a \in X \text{ and } a \notin Y\}$, and let $\overline{X} = \omega \backslash X$. Let $X \oplus Y$ be the set $Z \subseteq \omega$ defined by $Z(2n) = X(n)$ and $Z(2n+1) = Y(n)$.

(16) Suppose that $X, Y \subseteq \omega$. We write $X \subseteq^* Y$ to mean that $X \backslash Y$ is finite. We write $X =^* Y$ to mean that $X \subseteq^* Y$ and $Y \subseteq^* X$.

(17) Given $Z \subseteq \omega$ and $n \geq 1$, we let $[Z]^n = \{x \subseteq Z : |x| = n\}$.

(18) If $x \subseteq \omega$ is finite and $a \in \omega$, we write $x < a$ if a is greater than every element of x. If $x \subseteq \omega$ is finite and $Z \subseteq \omega$ is nonempty, we write $x < Z$ if every element of x is less than every element of Z.

(19) Suppose that $n \geq 1$, B is infinite, $f \colon [B]^{n+1} \to \omega$, $x \in [B]^n$, and $a \in B$. When we write $f(x, a)$, we implicitly assume that $x < a$, and we let $f(x, a) = f(x \cup \{a\})$. Also, if $n = 1$ and $a, b \in B$, when we write $f(a, b)$, we implicitly assume that $a < b$, and we let $f(a, b) = f(\{a, b\})$.

(20) Let $\langle \cdot \rangle$ denote a fixed effective bijective function from $\omega^{<\omega}$ to ω.

(21) Let $c, i \in \omega$. Fix $a_0, a_1, \ldots, a_{n-1} \in \omega$ such that $c = \langle a_0, a_1, \ldots, a_{n-1} \rangle$. If $i < n$, we let $(c)_i = a_i$, and if $i \geq n$, we let $(c)_i = 0$.

(22) Let $\{\varphi_e\}_{e \in \omega}$ be an effective listing of the partial Turing functionals.

(23) $\mathbf{a}$,$\mathbf{b}$,$\mathbf{c}$,... will denote Turing degrees.

(24) If $X \subseteq \omega$, we denote the Turing degree of X by $\deg(X)$.

(25) Let λ be Lebesgue measure on 2^ω.

(26) Given an infinite set A, let $p_A \colon \omega \to \omega$ be the function defined by letting $p_A(n)$ be the $(n+1)^{\text{st}}$ element of A in increasing order.

We make use of the following ordering in several constructions.

Definition 1.2.2. For each $n \in \omega$, we define a total ordering $<_n$ of $[\omega]^n$ as follows. For $x, y \in [\omega]^n$, we let $x <_n y$ if and only if $x \neq y$ and $x(i) < y(i)$, where i is the greatest integer less than n with $x(i) \neq y(i)$.

Definition 1.2.3. Given $p \geq 1$, let $\pi_1 \colon \omega \times p \to \omega$ denote the projection onto the first coordinate and $\pi_2 \colon \omega \times p \to p$ denote projection onto the second coordinate.

Chapter 2

Background and Motivation

König's Lemma and Ramsey's Theorem stand out as two of the most important and far-reaching combinatorial principles about ω. There has been an extensive study of the strength of these combinatorial principles using the tools of computability theory and reverse mathematics. From the viewpoint of computability theory (see [29] for the necessary background information about computability theory), one may ask where solutions to computable instances of these problems lie either in the Turing degrees or the arithmetical hierarchy. Also, one may seek to classify the strength of these statements with respect to the reverse mathematics hierarchy (see [28] for the necessary background information about reverse mathematics). Before embarking on our analysis of various partition theorems on ω, we will discuss some of the known results for König's Lemma and Ramsey's Theorem.

2.1 Effective Analysis of König's Lemma

To analyze König's Lemma, we first need a way to talk about trees. For our purposes, it is important to distinguish between arbitrary bounded trees and computably bounded trees.

Definition 2.1.1.

(1) A *tree* is a subset T of $\omega^{<\omega}$ such that for all $\sigma \in T$, if $\tau \in \omega^{<\omega}$ and $\tau \subseteq \sigma$, then $\tau \in T$.

(2) If T is a tree and $S \subseteq T$ is also a tree, we say that S is a *subtree* of T.

(3) A tree T is *bounded* if there exists $h\colon \omega \to \omega$ such that for all $\sigma \in T$ and $k \in \omega$ with $|\sigma| > k$, we have $\sigma(k) \leq h(k)$. If there exists such an h which is computable, we say that T is *computably bounded.*

(4) A *branch* of a tree T is a function $f\colon \omega \to \omega$ such that $f \restriction n \in T$ for all $n \in \omega$.

Theorem 2.1.2 (König's Lemma). *Every infinite bounded tree has a branch.*

Proof. Let T be an infinite tree and let $h\colon \omega \to \omega$ be such that $\sigma(k) \le h(k)$ whenever $\sigma \in T$ and $|\sigma| > k$. We define a sequence $\sigma_0 \subsetneq \sigma_1 \subsetneq \sigma_2 \subsetneq \dots$ inductively such that $\sigma_k \in \omega^{<\omega}$ and $|\sigma_k| = k$ for all $k \in \omega$. We maintain the invariant that $\{\tau \in T : \sigma_k \subseteq \tau\}$ is infinite. Let $\sigma_0 = \epsilon$ be the empty string. Suppose that we have defined σ_k and that $\{\tau \in T : \sigma_k \subseteq \tau\}$ is infinite. If $\{\tau \in T : \sigma_k * a \subseteq \tau\}$ is finite for all $a \le h(k)$, then $\{\tau \in T : \sigma_k \subseteq \tau\} = \{\sigma_k\} \cup \bigcup_{a \le h(k)} \{\tau \in T : \sigma_k * a \subseteq \tau\}$ is finite, a contradiction. Thus, we may let $\sigma_{k+1} = \sigma_k * a$ for the least $a \le h(k)$ such that $\{\tau \in T : \sigma_k * a \subseteq \tau\}$ is infinite.

Define $f\colon \omega \to \omega$ be letting $f(k) = \sigma_{k+1}(k)$. Then f is a branch of T. □

An effective analysis of König's Lemma depends on both the complexity of f and the complexity of the bound. We will mostly be concerned with subtrees of $2^{<\omega}$ (that is, trees which are bounded by $h(k) = 1$). It is straightforward to effectively code computably bounded computable trees by computable subtrees of $2^{<\omega}$, so for our purposes there is no loss in restricting attention to the following case.

Corollary 2.1.3 (Weak König's Lemma). *Every infinite subtree of* $2^{<\omega}$ *has a branch.*

A simple analysis of the proof of König's Lemma shows that every computable infinite subtree of $2^{<\omega}$ has a $\mathbf{0}''$-computable branch. However, instead of asking whether the sets $\{\tau \in T : \sigma_k * a \subseteq \tau\}$ are infinite (a 2-quantifier question), we may ask whether $\{\tau \in T : \sigma_k * a \subseteq \tau \text{ and } |\tau| = m\}$ is nonempty for every m (a 1-quantifier question due to the bound). This leads to the following:

Theorem 2.1.4 (Kreisel [18]). *Every computable infinite subtree of* $2^{<\omega}$ *has a* $\mathbf{0}'$*-computable branch. Hence, every computable infinite subtree of* $2^{<\omega}$ *has a* Δ^0_2 *branch.*

Definition 2.1.5. Let $\mathbf{a}$ and $\mathbf{b}$ be Turing degrees. We write $\mathbf{a} \gg \mathbf{b}$ to mean that every infinite $\mathbf{b}$-computable subtree of $2^{<\omega}$ has an $\mathbf{a}$-computable branch.

The notation $\mathbf{a} \gg \mathbf{b}$ was introduced in Simpson [27], and many of the basic properties of this ordering can be found there. The following well-known lemma gives some equivalent characterizations of this ordering.

Lemma 2.1.6 (Scott [25], Solovay). *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be Turing degrees. The following are equivalent:*

(1) $\mathbf{a} \gg \mathbf{b}$

(2) Every partial $\{0,1\}$*-valued* $\mathbf{b}$*-computable function has a total* $\mathbf{a}$*-computable extension.*

(3) $\mathbf{a}$ *is the degree of a complete extension of the theory of Peano Arithmetic with an additional unary predicate symbol* P*, axioms* $P(n)$ *for all* $n \in B$ *and* $\neg P(n)$ *for all* $n \notin B$ *(where* B *is a fixed set in* $\mathbf{b}$*), and induction axioms for formulas involving* P*.*

Using the existence of a computable tree in which the branches code complete extensions of Peano Arithmetic, it follows that there is a "universal" computable subtree of $2^{<\omega}$.

Corollary 2.1.7. *There exists an infinite computable subtree T of $2^{<\omega}$ such that given any branch B_T of T, and any infinite computable subtree S of $2^{<\omega}$, there exists a branch B_S of S such that $B_S \leq_T B_T$.*

It has long been known that there is a computable infinite subtree of $2^{<\omega}$ with no computable branch. Relativizing this and Theorem 2.1.4, it follows that $\mathbf{a} \geq \mathbf{b}' \rightarrow \mathbf{a} \gg \mathbf{b} \rightarrow \mathbf{a} > \mathbf{b}$. In [15], Jockusch and Soare established the following strengthening of Theorem 2.1.4.

Theorem 2.1.8 (Low Basis Theorem [15, Theorem 2.1]). *There exists $\mathbf{a} \gg \mathbf{0}$ with $\mathbf{a}' = \mathbf{0}'$.*

Jockusch and Soare also established the following cone avoidance theorem.

Theorem 2.1.9 (Jockusch and Soare [15, Theorem 2.5]). *Suppose that $\{\mathbf{b}_k\}_{k \in \omega}$ is a family of nonzero degrees. There exists $\mathbf{a} \gg \mathbf{0}$ such that $\mathbf{b}_k \not\leq \mathbf{a}$ for all $k \in \omega$.*

In terms of reverse mathematics, the existence of a computable infinite subtree of $2^{<\omega}$ with no computable branch implies that Weak König's Lemma is not provable in RCA_0. Combining the axioms of RCA_0 with Weak König's Lemma gives the important system WKL_0.

2.2 Effective Analysis of Ramsey's Theorem

Definition 2.2.1.

(1) Suppose that $n, p \geq 1$, $B \subseteq \omega$ is infinite, and $f \colon [B]^n \rightarrow p$. Such an f is called a *p-coloring* of $[B]^n$ and n is called the *exponent*. We say that a set $H \subseteq B$ is *homogeneous* for f if H is infinite and $f(x) = f(y)$ for all $x, y \in [B]^n$.

(2) Suppose that $n, p \geq 1$, $B \subseteq \omega$ is infinite, and $f \colon [B]^{n+1} \rightarrow p$. We say that a pair (A, g), where $A \subseteq B$ is infinite and $g \colon [A]^n \rightarrow p$, is a *prehomogeneous pair* for f if $f(x, a) = g(x)$ for all $x \in [A]^n$ and all $a \in A$ with $x < a$.

Theorem 2.2.2 (Ramsey's Theorem [23]). *Suppose that $n, p \geq 1$, $B \subseteq \omega$ is infinite, and $f \colon [B]^n \rightarrow p$. There exists a set homogeneous for f.*

Specker [30] was the first to analyze the effective content of Ramsey's Theorem, and he showed that there exists a computable $f \colon [\omega]^2 \rightarrow 2$ with no computable homogeneous set. Before discussing further bounds on the complexity of homogeneous sets, we first give a few proofs of Ramsey's Theorem.

Our proofs of Ramsey's Theorem break down into the following three steps, and differ only in their proofs of (1):

(1) Given $f\colon [B]^{n+1} \to p$, construct a prehomogeneous pair (A, g) for f.

(2) Apply induction to $g\colon [A]^n \to p$.

(3) Show that any set homogeneous for g is homogeneous for f.

Before proving the existence of prehomogeneous pairs, we first establish (3) to verify their utility.

Claim 2.2.3. *Suppose that $n, p \geq 1$, $B \subseteq \omega$ is infinite, $f\colon [B]^{n+1} \to p$, and (A, g) is a prehomogeneous pair for f. If $H \subseteq A$ is homogeneous for g, then H is homogeneous for f.*

Proof. Let $H \subseteq A$ be homogeneous for g. Let $x, y \in [H]^n$ and $a, b \in H$ with $x < a$ and $y < b$. We have $f(x, a) = g(x) = g(y) = f(y, b)$, hence H is homogeneous for f. □

We next prove that prehomogeneous pairs exist. We give three proofs, because each of the techniques are relevant for later constructions, and each can be used to provide a different computability-theoretic analysis. To facilitate the constructions, we first define a notion of prehomogeneous triple which will provide an approximation to a desired prehomogeneous pair.

Definition 2.2.4. Suppose that $n, p \geq 1$, $B \subseteq \omega$ is infinite, and $f\colon [B]^{n+1} \to p$. We call a triple (z, I, g) where $z \subseteq B$ is finite, $I \subseteq B$ is infinite, $z < I$, and $g\colon [z]^n \to p$, a *prehomogeneous triple* for f if for all $x \in [z]^n$ and all $a \in z \cup I$ with $x < a$, we have $f(x, a) = g(x)$.

Proposition 2.2.5. *Suppose that $n, p \geq 1$, $B \subseteq \omega$ is infinite, and $f\colon [B]^{n+1} \to p$. There exists a prehomogeneous pair (A, g) for f.*

Proof 1. This proof is the most straightforward, and proceeds by repeatedly thinning down a set of candidates to add to the prehomogeneous pair, while ensuring that this set of candidates is always infinite. We inductively define a sequence $(a_m, I_m, g_m)_{m \in \omega}$ such that

- $a_m \in B$.
- $I_m \subseteq B$ is infinite.
- $g_m\colon [\{a_i : i \leq m\}]^n \to p$.
- $a_0 < a_1 < \cdots < a_{m-1} < a_m < I_m \subseteq I_{m-1} \subseteq \cdots \subseteq I_1 \subseteq I_0$.
- $g_0 \subseteq g_1 \subseteq \cdots \subseteq g_m$.

- $(\{a_i : i \leq m\}, I_m, g_m)$ is a prehomogeneous triple for f.

We begin by letting $a_{-1} = -1$, $I_{-1} = B$, and $g_{-1} = \emptyset$.

Suppose that we have defined our sequence through stage $m \geq -1$. We first let $a_{m+1} = \min(I_m)$, $I'_m = I_m - \{a_{m+1}\}$, and $g_{m+1}(x) = g_m(x)$ for all $x \in [\{a_i : i \leq m\}]^n$. List the elements of $[\{a_i : i \leq m+1\}]^n$ whose greatest element is a_{m+1} as $x_0, x_1, \ldots, x_{l-1}$, where $l = \binom{m+1}{n-1}$. If $l = 0$, let $I_{m+1} = I'_m$ and $g_{m+1} = g_m$. Otherwise, we proceed inductively through the x_k, defining $g_{m+1}(x_k)$ and infinite sets $J_0, J_1, \ldots, J_l$ such that $I'_m = J_0 \supseteq J_1 \supseteq \cdots \supseteq J_l$ along the way. Let $J_0 = I'_m$. Suppose that $k < l$, and we have defined J_k. Since J_k is infinite, there exists $q < p$ such that there are infinitely many $b \in J_k$ with $f(x_k, b) = q$. Fix the least such q, and let $g_{m+1}(x_k) = q$ and $J_{k+1} = \{b \in J_k : f(x_k, b) = q\}$. Proceed to the next value of $k < l$, if it exists.

Once J_l has been defined, let $I_{m+1} = J_l$. One easily checks that the invariants are maintained. This completes stage $m + 1$.

Finally, let $A = \{a_m : m \in \omega\}$ and $g = \bigcup_{m \in \omega} g_m$. Then (A, g) is a prehomogeneous pair for f. $\square$

Proof 2. This proof makes use of an infinite bounded tree T in which the branches code prehomogeneous pairs. Let $B = \{b_0 < b_1 < b_2 < \ldots\}$. We first give an intuitive picture of the construction. We inductively place the elements of B on a tree ordered by $\prec$ in the following manner. To begin, put b_0,b_1,…,b_{n-1} on a tree with b_0 at the root and $b_0 \prec b_1 \prec \cdots \prec b_{n-1}$. Suppose that we've placed b_i for all $i < m$. We now work our way up the tree to place b_m as a new leaf. Start at node b_{n-1}. If we're at node b_j, look to see if there is an immediate $\prec$-successor b_k to b_j such that $f(x, b_k) = f(x, b_m)$ for all $x \in [\{b_l : b_l \preceq b_j\}]^n$ whose last element is b_j. If so, move to node b_k and continue up the tree. Otherwise, place b_m as a new leaf with $b_j \prec b_m$. Notice that a branch of this tree will code a prehomogeneous pair.

More precisely, we define a sequence $\{T_m\}_{m \in \omega}$ of trees and a sequence $\{\ell_m\}_{m \in \omega}$ of functions such that

- $T_m \subseteq \omega^{<\omega}$.
- $T_0 \subseteq T_1 \subseteq T_2 \subseteq \ldots$.
- $|T_m| = m$.
- $\ell_m : T_m \to B$ with $\mathrm{range}(\ell_m) = \{b_i : i < m\}$
- $\ell_0 \subseteq \ell_1 \subseteq \ell_2 \subseteq \ldots$

Given $\sigma \in T_m$, we think of $\ell_m(\sigma)$ as a label from B for the node $\sigma \in T_m$. We initially define T_m and ℓ_m for all $m \leq n$. For $k \in \omega$, let 0^k be the string of k zeros. Given $m \leq n$, let $T_m = \{0^k : k < m\}$ and let $\ell_m(0^k) = b_k$ for all $k < m$.

Suppose that $m \geq n$, and we have defined T_m and ℓ_m. Let $\ell_{m+1}(\tau) = \ell_m(\tau)$ for all $\tau \in T_m$. Define a sequence of elements of T_m inductively as follows. For k with $0 \leq k < n$, let $\sigma_k = 0^k$. Suppose that $k \geq n-1$ and we have defined σ_i for all $i \leq k$. List the elements of $[\ell_m(\sigma_0), \ell_m(\sigma_1), \ldots, \ell_m(\sigma_k)]^n$ whose last element is $\ell_m(\sigma_k)$ as $x_0 <_n x_1 <_n \cdots <_n x_{l-1}$ (recall Definition 1.2.2) where $l = \binom{k}{n-1}$. Let $c = \langle f(x_0, b_m), f(x_1, b_m), \ldots, f(x_{l-1}, b_m)\rangle$. If $\sigma_k * c \in T_m$, let $\sigma_{k+1} = \sigma_k * c$ and continue the induction. Otherwise, let $T_{m+1} = T_m \cup \{\sigma_k * c\}$, and let $\ell_{m+1}(\sigma_k * c) = b_m$. Notice that the induction must stop at some finite stage because $|T_m| = m$.

Let $T = \bigcup_{m\in\omega} T_m$ and $\ell = \bigcup_{m\in\omega} \ell_m$. Now T is an infinite tree and is bounded by $h\colon \omega \to \omega$ defined by $h(k) = \max(\{0\} \cup \{\langle a_0, a_1, \ldots, a_{l-1}\rangle : l = \binom{k}{n-1}$ and $a_i < p$ whenever $1 \leq i \leq l\})$. By König's Lemma, T has a branch f. For each $k \in \omega$, let $\sigma_k = f \upharpoonright k$. Let $A = \{\ell(\sigma_k) : k \in \omega\}$ and define $g\colon [A]^n \to p$ as follows. Suppose that $k_1 < k_2 < \cdots < k_n$ in ω. List the elements of $[\ell(\sigma_0), \ell(\sigma_1), \ell(\sigma_2), \ldots, \ell(\sigma_{k_n})]^n$ whose last element is $\ell(\sigma_{k_n})$ as $x_0 <_n x_1 <_n \cdots <_n x_{l-1}$, and fix $i < l$ such that $x_i = \{\ell(\sigma_{k_1}), \ell(\sigma_{k_2}), \ldots, \ell(\sigma_{k_n})\}$. Let $g(\ell(\sigma_{k_1}), \ell(\sigma_{k_2}), \ldots, \ell(\sigma_{k_n})) = (\sigma_{k_n+1})_i$. Then (A, g) is a prehomogeneous pair for f. $\square$

Proof 3. This proof is similar to the first, but we make use of an ultrafilter to guide our inductive construction. Let $\mathcal{U}$ be a nonprincipal ultrafilter on ω with $B \in \mathcal{U}$, i.e. $\mathcal{U} \subseteq \mathcal{P}(\omega)$ such that

- $B \in \mathcal{U}$ and $\emptyset \notin \mathcal{U}$.
- For all $X, Y \in \mathcal{P}(\omega)$, if $X \in \mathcal{U}$ and $X \subseteq Y$, then $Y \in \mathcal{U}$.
- For all $X, Y \in \mathcal{U}$, we have $X \cap Y \in \mathcal{U}$.
- For all $X \in \mathcal{P}(\omega)$, either $X \in \mathcal{U}$ or $\overline{X} \in \mathcal{U}$.
- For all cofinite $X \in \mathcal{P}(\omega)$, we have $X \in \mathcal{U}$.

We inductively define a sequence $(a_m, I_m, g_m)_{m\in\omega}$ such that

- $a_m \in B$.
- $I_m \subseteq B$ and $I_m \in \mathcal{U}$.
- $g_m\colon [\{a_i : i \leq m\}]^n \to p$.
- $a_0 < a_1 < \cdots < a_{m-1} < a_m < I_m \subseteq I_{m-1} \subseteq \cdots \subseteq I_1 \subseteq I_0$.
- $g_0 \subseteq g_1 \subseteq \cdots \subseteq g_m$.
- $(\{a_i : i \leq m\}, I_m, g_m)$ is a prehomogeneous triple for f.

We begin by letting $a_{-1} = -1$, $I_{-1} = B$, and $g_{-1} = \emptyset$.

Suppose that we have defined our sequence through stage $m \geq -1$. We first let $a_{m+1} = \min(I_m)$, $I'_m = I_m - \{a_{m+1}\}$, and $g_{m+1}(x) = g_m(x)$ for all $x \in [\{a_i : i \leq m\}]^n$. Let $\mathcal{F}$ be the set of those elements of $[\{a_i : i \leq m+1\}]^n$ whose last element is a_{m+1}. If $\mathcal{F} = \emptyset$, let $I_{m+1} = I'_m$ and $g_{m+1} = g_m$. Suppose $\mathcal{F} \neq \emptyset$. For each $x \in \mathcal{F}$, there is exactly one $q < p$ such that $Z_{x,q} = \{b \in I'_m : f(x,p) = q\} \in \mathcal{U}$ (since $I'_m \in \mathcal{U}$, $\bigcup_{q<p} Z_{x,q} = I'_m$, and $Z_{x,q} \cap Z_{x,r} = \emptyset$ whenever $q, r < p$ with $q \neq r$). For each $x \in \mathcal{F}$, let q_x be the unique q with $Z_{x,q} \in \mathcal{U}$. Let $I_{m+1} = \bigcap_{x \in \mathcal{F}} Z_{x,q_x}$, and notice that $I_{m+1} \in \mathcal{U}$. For each $x \in \mathcal{F}$, let $g_{m+1}(x) = q_x$. One easily checks that $(\{a_i : i \leq m+1\}, I_{m+1}, g_{m+1})$ is a prehomogeneous triple for f.

Finally, let $A = \{a_m : m \in \omega\}$ and $g = \bigcup_{m \in \omega} g_m$. Then (A, g) is a prehomogeneous pair for f. $\square$

We finally give a proof of Ramsey's Theorem.

Proof of Theorem 2.2.2. The proof is by induction on n. Suppose that $n = 1$ so that we have $f\colon [B]^1 \to p$. Since B is infinite and $p \in \omega$, there exists $q < p$ such that the set $A_q = \{b \in B : f(b) = q\}$ is infinite. Notice that A_q is homogeneous for any such q.

Suppose that the theorem holds for n, and we are given $f\colon [B]^{n+1} \to p$. By Proposition 2.2.5, there exists a prehomogeneous pair (A, g) for f. Applying the inductive hypothesis to g, there exists $H \subseteq A$ which is homogeneous for g. By Claim 2.2.3, H is homogeneous for f. $\square$

Examining the above proofs, we see that the key feature in analyzing the computability-theoretic complexity of homogeneous sets resides in the complexity of prehomogeneous pairs for computable f. A simple analysis of the first proof shows that if B and f are computable, then there is a prehomogeneous pair (A, g) for f such that $\deg(A \oplus g) \leq \mathbf{0}''$. By applying induction and relativizing, we get the result that if $B \subseteq \omega$ is infinite and computable, and $f\colon [B]^n \to p$ is computable, then there is a homogeneous set H for f with $\deg(H) \leq \mathbf{0}^{(2n-2)}$. However, an analysis of the second proof shows that if B and f are computable, then the tree T constructed is $\mathbf{0}'$-computable and computably bounded, so we get the following better result (using the Low Basis Theorem relative to $\mathbf{0}'$ for the last statement).

Proposition 2.2.6. *Suppose that $n, p \geq 1$, $B \subseteq \omega$ is infinite and computable, $f\colon [B]^{n+1} \to p$ is computable, and $\mathbf{a} \gg \mathbf{0}'$. There exists a prehomogeneous pair (A, g) for f such that deg$(A \oplus g) \leq \mathbf{a}$. In particular, there exists a prehomogeneous pair (A, g) for f such that deg$(A \oplus g)' \leq \mathbf{0}''$*

If we apply induction and relativize, we conclude the following.

Theorem 2.2.7 (Jockusch [12, Theorem 5.6]). *Suppose that $n \geq 2$, $p \geq 1$, $B \subseteq \omega$ is infinite and computable, $f\colon [\omega]^n \to p$ is computable, and $\mathbf{a} \gg \mathbf{0}^{(n-1)}$. There exists a homogeneous set H for f such that deg$(H) \leq \mathbf{a}$.*

To analyze the third proof, we need to replace the ultrafilter by another object which plays a similar role.

Definition 2.2.8. A set $V \subseteq \omega$ is *r-cohesive* if V is infinite and for every computable set Z, either $V \cap Z$ is finite or $V \cap \overline{Z}$ is finite.

The following easy claim establishes the usefulness of r-cohesive sets in our context, by characterizing them as the infinite sets which are homogeneous, modulo finite sets, for all computable $f\colon [\omega]^1 \to p$.

Claim 2.2.9. *Let V be an infinite set. The following are equivalent:*

(1) V is r-cohesive.

(2) For every computable $f\colon [\omega]^1 \to 2$, there exists a finite $z \subseteq \omega$ such that $V \backslash z$ is homogeneous for f.

(3) For every $p \geq 2$ and every computable $f\colon [\omega]^1 \to p$, there exists a finite $z \subseteq \omega$ such that $V \backslash z$ is homogeneous for f.

To see how r-cohesive sets can play a role analogous to the ultrafilter in Proof 3 above, suppose that $V \subseteq B$ is r-cohesive and $f\colon [B]^{n+1} \to p$ is computable. Given $x \in [B]^n$ and $q < p$, the set $Z_{x,q} = \{b \in B : f(x,b) = q\}$ is computable, so either either $V \cap Z_{x,q}$ is finite or $V \cap \overline{Z_{x,q}}$ is finite. Applying this for each $q < p$, it follows that there exists a unique $q < p$ such that $V \cap \overline{Z_{x,q}}$ is finite. Thus, the r-cohesive set guides our inductive construction like the ultrafilter does in Proof 3 of Theorem 2.2.5.

Jockusch and Stephan [9] (see also [10] for a correction) characterized the Turing degrees of jumps of r-cohesive sets.

Theorem 2.2.10 (Jockusch and Stephan [9, Theorem 2.2(ii)]). *Suppose that* $\mathbf{a} \gg \mathbf{0}'$. *There exists an r-cohesive set V such that deg$(V)' \leq \mathbf{a}$. Furthermore, every r-cohesive set V satisfies deg$(V)' \gg \mathbf{0}'$.*

Using this result and a suitable r-cohesive set in place of the ultrafilter in Proof 3 above gives another proof of Proposition 2.2.6. However, by a much more detailed analysis of this approach when $n = 2$, Cholak, Jockusch, and Slaman improved this by a jump.

Theorem 2.2.11 (Cholak, Jockusch, Slaman [1, essentially Lemma 4.6]). *Suppose $p \geq 1$, $B \subseteq \omega$ is infinite and computable, $f\colon [B]^2 \to p$ is computable, and* $\mathbf{a} \gg \mathbf{0}'$. *There exists a homogeneous set H for f such that deg$(H)' \leq \mathbf{a}$.*

In an interesting reversal of the role of cohesiveness, Cholak, Jockusch, and Slaman also established that there exists a computable $f\colon [\omega]^2 \to 2$ such that all homogeneous sets satisfy a cohesiveness property slightly weaker than that for r-cohesive sets.

Definition 2.2.12. A set $V \subseteq \omega$ is *p-cohesive* if V is infinite and for every primitive recursive set Z, either $V \cap Z$ is finite or $V \cap \overline{Z}$ is finite.

Proposition 2.2.13 (Cholak, Jockusch, Slaman [1, Theorem 12.5]). *There exists a computable $f\colon [\omega]^2 \to 2$ such that every set homogeneous for f is p-cohesive.*

Jockusch and Stephan [9] also characterized the Turing degrees of p-cohesive sets.

Theorem 2.2.14 (Jockusch and Stephan [9, Theorem 2.1]). *For every degree $\mathbf{a}$, $\mathbf{a}$ is p-cohesive if and only if $\mathbf{a}' \gg \mathbf{0}'$.*

Combining Theorem 2.2.14 and Proposition 2.2.13, we get the following corollary showing that Theorem 2.2.11 is sharp.

Corollary 2.2.15. *There exists a computable $f\colon [\omega]^2 \to 2$ such that $deg(H)' \gg \mathbf{0}'$ for all sets H homogeneous for f.*

Therefore, as remarked on pp. 50-51 of [1], we get a corollary about Ramsey's Theorem for exponent 2 similar to Corollary 2.1.7 about König's Lemma with "jump universal" in place of "universal".

Corollary 2.2.16. *There exists a computable $f\colon [\omega]^2 \to 2$ such that that given any set H_f homogeneous for f, and any computable $g\colon [\omega]^2 \to 2$, there exists a set H_g homogeneous for g with $H_g' \leq_T H_f'$.*

Another fundamental step in understanding the strength of Ramsey's Theorem for exponent 2 was taken by Seetapun, who proved the following cone avoidance theorem, hence establishing that is impossible to code anything into homogeneous sets for a computable $f\colon [\omega]^2 \to p$.

Theorem 2.2.17 (Seetapun [26]). *Suppose that $p \geq 1$, $B \subseteq \omega$ is infinite and computable, $f\colon [B]^2 \to p$ is computable and $\{\mathbf{b}_k\}_{k\in\omega}$ is a family of nonzero degrees. There exists a set H homogeneous for f such that $\mathbf{b}_k \not\leq deg(H)$ for all $k \in \omega$.*

In contrast, the following two propositions show that the halting problem can be coded into the homogeneous sets of a computable $f\colon [\omega]^3 \to 2$ and a c.e. $f\colon [\omega]^2 \to 2$. They will play important roles in the coding techniques used throughout this dissertation.

Proposition 2.2.18 (Jockusch [12, Lemma 5.9]). *For every $n \geq 3$, there exists a computable $h\colon [\omega]^n \to 2$ such that for all sets H homogeneous for h, we have $h([H]^2) = \{0\}$ and $H \geq_T 0^{(n-2)}$.*

Proposition 2.2.19 (Jockusch and Hummel [8, Lemma 3.7]). *There exists a c.e. $h\colon [\omega]^2 \to 2$ (that is, $\{x \in [\omega]^2 : h(x) = 1\}$ is c.e.) such that for all sets H homogeneous for h, we have $h([H]^2) = \{0\}$ and $H \geq_T 0'$.*

We first show how we can use a relativization of Proposition 2.2.18 together with the Limit Lemma to lift results for exponent 2 to higher exponents. We state the theorem in relativized form to facilitate the inductive proof.

Proposition 2.2.20. *Suppose that $X \subseteq \omega$, $n \geq 2$, $p \geq 1$, $B \subseteq \omega$ is infinite and X-computable, $f\colon [B]^n \to p$ is X-computable, and $\mathbf{a} \gg deg(X)^{(n-1)}$. There exists a homogeneous set H for f such that $deg(H)' \leq \mathbf{a}$. Furthermore, for every $X \subseteq \omega$ and every $n \geq 2$, there exists an X-computable $f\colon [\omega]^n \to 3$ such that for all sets H homogeneous for f, we have $deg(H \oplus X) \geq deg(X)^{(n-2)}$ and $deg(H \oplus X)' \gg deg(X)^{(n-1)}$.*

Proof. We prove the first statement by induction on n. The case $n = 2$ follows by relativizing Theorem 2.2.11. Suppose that $n \geq 2$ and the result holds for n. Suppose that B and $f\colon [B]^{n+1} \to p$ are X-computable, and $\mathbf{a} \gg \deg(X)^{(n)}$. Relativizing Proposition 2.2.6 to X, there exists a prehomogeneous pair (A, g) for f with $\deg(A \oplus g)' \leq \deg(X)''$. By the inductive hypothesis, there exists a set H homogeneous for $g\colon [A]^n \to p$ with $\deg(H)' \leq \mathbf{a}$ since $\mathbf{a} \gg \deg(X)^{(n)} = \deg(X'')^{(n-2)} \geq (\deg(A \oplus g)')^{(n-2)} = \deg(A \oplus g)^{(n-1)}$. By Claim 2.2.3, H is homogeneous for f.

We prove the second part of the proposition in following strong form. For every $X \subseteq \omega$ and every $n \geq 2$, there exists an X-computable $f\colon [\omega]^n \to 3$ such that for all sets H homogeneous for f, we have $f([H]^n) \neq \{2\}$, $\deg(H \oplus X) \geq \deg(X)^{(n-2)}$ and $\deg(H \oplus X)' \gg \deg(X)^{(n-1)}$. The case $n = 2$ follows by relativizing Corollary 2.2.15. Suppose that $n \geq 2$ and the result holds for n. Fix an X'-computable $g\colon [\omega]^n \to 3$ such that for all sets H homogeneous for g, we have $g([H]^n) \neq \{2\}$, $\deg(H \oplus X') \geq \deg(X)^{(n-1)}$ and $\deg(H \oplus X')' \gg \deg(X)^{(n)}$. By the Limit Lemma, there is an X-computable $g_1\colon [\omega]^{n+1} \to 3$ such that $g(x) = \lim_s g_1(x, s)$ for all $x \in [\omega]^n$. Notice that if H homogeneous for g_1, then H is homogeneous for g. By Proposition 2.2.18 relativized to X and the fact that $n+1 \geq 3$, there exists an X-computable $h\colon [\omega]^{n+1} \to 2$ such that for all infinite sets H homogeneous for h, we have $h([H]^2) = \{0\}$ and $H \oplus X \geq_T X'$. Define an X-computable $f\colon [\omega]^{n+1} \to 3$ by

$$f(y) = \begin{cases} g_1(y) & \text{if } h(y) = 0 \\ 2 & \text{if } h(y) = 1 \end{cases}$$

Suppose that H is homogeneous for f. If $f([H]^{n+1}) = \{2\}$, then for all $y \in [H]^{n+1}$, either $h(y) = 1$ or $g_1(y) = 2$. By Ramsey's Theorem applied to the function $h \restriction [H]^{n+1} : [H]^{n+1} \to 2$, there exists an infinite set $I \subseteq H$ such that either $h([I]^{n+1}) = \{1\}$ or $h([I]^{n+1}) = \{0\}$, and hence $g_1([I]^{n+1}) = \{2\}$, both of which are impossible. Therefore, $f([H]^{n+1}) \neq \{2\}$, and hence H is homogeneous for both h and g_1. Since H is homogeneous for h, we have $H \oplus X \geq_T X'$. Since every set homogeneous for g_1 is also homogeneous for g, we have $g_1([H]^n) \neq \{2\}$, $\deg(H \oplus X') \geq \deg(X)^{(n-1)}$ and $\deg(H \oplus X')' \gg \deg(X)^{(n)}$. Hence, $f([H]^{n+1}) \neq \{2\}$,

$\deg(H \oplus X) \geq \deg(H \oplus X') \geq \deg(X)^{(n-1)}$ and $\deg(H \oplus X)' \geq \deg(H \oplus X')' \gg \deg(X)^{(n)}$. □

The following question of whether we can replace the 3-coloring from the previous proposition by a 2-coloring is open.

Question 2.2.21. *For each $n \geq 3$, does there exist a computable $f\colon [\omega]^n \to 2$ such that for all sets H homogeneous for f, we have $deg(H) \geq \mathbf{0}^{(n-2)}$ and $deg(H)' \gg \mathbf{0}^{(n-1)}$?*

Jockusch also characterized the location of homogeneous sets for computable f in the arithmetical hierarchy.

Theorem 2.2.22 (Jockusch [12, Theorem 5.1, Theorem 5.5]). *Suppose that $n, p \geq 2$, $B \subseteq \omega$ is infinite and computable, and $f\colon [B]^n \to p$ is computable. There exists a Π^0_n homogeneous set for f. Furthermore, for each $n \geq 2$, there exists a computable $f\colon [\omega]^n \to 2$ with no Σ^0_n set homogeneous for f.*

We give a proof of the first part of this theorem in the case $n = 2$, as it will be useful to refer to when we generalize it in Chapter 3.

Proof. Suppose that $n = 2$, $p \geq 2$, B is computable, and $f\colon [B]^2 \to p$ is computable. We seek to build a prehomogeneous pair (A, g) such that $A = \{a_0 < a_1 < a_2 < \dots\}$ is Π^0_2 by using a $0'$-oracle to enumerate its complement. We imitate Proof 1 of Theorem 2.2.5 above, but we must be careful to avoid the 2-quantifier question of whether certain sets are infinite. Hence, when building our set, we ask only 1-quantifier questions which give approximate answers to whether certain sets are infinite. Of course, we can be misled, but our approximations will eventually be correct and we can discard progress that has been made along tainted paths.

The construction is a movable marker construction using a $0'$-oracle. We denote by a^s_i the position of the $(i+1)^{\text{st}}$ marker Λ_i at the beginning of stage s. At the beginning of each stage s, we will have a number n^s such that the markers currently having a position are exactly the Λ_i for $i < n^s$, and for each $i < n^s$, we will have a number q^s_i, representing the current approximation to $g(a^s_i)$. Let β^s be the greatest position of any marker up to stage s ($\beta^s = 0$ if $s = 0$). Given these and $k \leq n^s$, we say that a number b is *k-acceptable* at s if

- $b \in B$.
- $b > \beta^s$.
- For all $i < k$, $f(a^s_i, b) = q^s_i$.

Construction: First set $n^0 = 0$. Stage $s \geq 0$: Assume inductively that we have n^s such that the markers currently having a position are exactly the Λ_i for $i < n^s$, along with q_i^s for all $i < n^s$. Enumerate into $\overline{A}$ all numbers $b \leq \beta^s$ such that $b \neq a_i^s$ for all $i < n^s$. Using a $0'$-oracle, let k^s be the largest $k \leq n^s$ such that there exists a number which is k-acceptable at s. Note that k^s exists because every sufficiently large element of B is 0-acceptable at s.

Case 1: *$k^s = n^s$:* Set $n^{s+1} = n^s + 1$ and place the marker Λ_{n^s} on the least k^s-acceptable number. Leave all markers Λ_i with $i < n^s$ in place, and let $q_i^{s+1} = q_i^s$ for all $i < n^s$. Also, let $q_{n^s}^{s+1} = 0$. (Place a new marker, and give it the first color.)

Case 2: *$k^s < n^s$:* Set $n^{s+1} = k^s + 1$ and detach all markers Λ_i with $k^s < i < n^s$. Leave all markers Λ_i with $i \leq k^s$ in place and let $q_i^{s+1} = q_i^s$ for all $i < k^s$. Also, let $q_{k^s}^{s+1} = q_{k^s}^s + 1$. (Discard mistakes and move to the next color.)

End Construction.

Claim 2.2.23. *For all $k \in \omega$, each limit $\lim_s a_k^s$ and $\lim_s q_k^s$ exists, so we may define $a_k = \lim_s a_k^s$ and $q_k = \lim_s q_k^s$. Furthermore $q_k < p$ for all $k \in \omega$.*

Proof. We proceed by induction. We assume that the Claim is true for all $i < k$ and prove it for k. Let t be the least stage such that for all $i < k$ and all $s \geq t$, we have $a_i^s = a_i$, and $q_i^s = q_i$. At stage t, the marker Λ_k is placed on a number b via Case 1 of the construction, so $n^{t+1} = k + 1$. Since each of a_i^s and q_i^s for $i < k$ have come to their limits, we must have $k^s \geq k$ and hence $n^s \geq k+1$ for all $s > t$ by construction (because if $s > t$ is least such that $k^s < k$, then we enter Case 2, so $q_{k^s}^s$ changes). Therefore, by construction, we never again move the marker Λ_k, so $a_k^s = a_k^{t+1}$ for all $s \geq t + 1$ and we may let $a_k = \lim_s a_k^s$.

We now show that $\lim_s q_k^s$ exists by showing that there are at most $p - 1$ stages $s > t$ such that $k^s = k$. This suffices, because q_k^s increases by 1 only at such s. Suppose then that there exists p stages $s > t$ such that $k^s = k$. For each $q < p$, there is a unique $s_q > t$ such that $q_k^{s_q} = q$, and $k^{s_q} = k$. Given any $q < p$, since $k^{s_q} = k$, there are no numbers which are $(k + 1)$-acceptable at s_q. Let $r = \max\{s_q : q < p\}$, and let b be the least number which is k-acceptable at r (such a number exists because otherwise we have $k^r < k$, which we know is not true). If we let $q = f(a_k, b)$, then b is $(k + 1)$-acceptable at s_q (since $\beta^{s_q} \leq \beta^r$ as $s_q \leq r$), contradicting the fact that no number is $(k + 1)$-acceptable at s_q. It follows that there are at most $p - 1$ stages $s > t$ such that $k^s = k$, so the proof of the Claim is complete. □

Claim 2.2.24. *Let $q < p$ be least such that $\{k : q_k = q\}$ is infinite. Then $H = \{a_k : q_k = q\}$ is a Π^0_2 homogeneous set for f.*

Proof. To see that H is Π^0_2, perform the above construction, with the additional action of enumerating the number $a^s_{k^s}$ at stage s if $q^s_{k^s} > q$. Then a_k is not enumerated if and only if $q_k \leq q$. Since $\{a_k : q_k < q\}$ is finite, it follows (by removing this finite set) that H is Π^0_2. Suppose that $i < j$ and $a_i, a_j \in H$. Let s be least such that $a^s_j = a_j$. By construction, a_j was $(i+1)$-acceptable at s, hence $f(a_i, a_j) = q_i = q$. It follows that H is a Π^0_2 homogeneous set for f. □

□

In terms of reverse mathematics, for each exponent $n \geq 3$, Ramsey's Theorem for exponent n is equivalent to ACA_0 over RCA_0 (see [28]). The reverse mathematical strength of Ramsey Theorem for exponent 2 is still somewhat mysterious, although [1] has considerably clarified the issue. It follows from the Low Basis Theorem and Theorem 2.2.22 that WKL_0 does not imply Ramsey's Theorem for exponent 2. By using Theorem 2.2.17, Seetapun proved that Ramsey's Theorem for exponent 2 does not imply ACA_0 over RCA_0, hence Ramsey's Theorem for exponent 2 is not equivalent to any of the usual systems of reverse mathematics over RCA_0.

2.3 The Canonical Ramsey Theorem

In Chapter 3, we study analogous questions for the Canonical Ramsey Theorem, a partition theorem due to Erdös and Rado about functions $f \colon [\omega]^n \to \omega$, i.e. colorings with infinitely many colors. Of course, we can not expect to always have homogeneous sets, as witnessed by the following simple functions $f \colon [\omega]^2 \to \omega$:

(1) $f(a, b) = a$

(2) $f(a, b) = b$

(3) $f(a, b) = \langle a, b \rangle$

However, the Canonical Ramsey Theorem for exponent 2 says that given any $f \colon [\omega]^2 \to \omega$, there exists an infinite set $C \subseteq \omega$ which either is homogeneous, or on which f behaves like one of the above functions. Precisely, given any $f \colon [\omega]^2 \to \omega$, there exists an infinite C such that either

(1) For all $a_1, b_1, a_2, b_2 \in C$ with $a_1 < b_1$ and $a_2 < b_2$, we have $f(a_1, b_1) = f(a_2, b_2)$.

(2) For all $a_1, b_1, a_2, b_2 \in C$ with $a_1 < b_1$ and $a_2 < b_2$, we have $f(a_1, b_1) = f(a_2, b_2) \leftrightarrow a_1 = a_2$.

(3) For all $a_1, b_1, a_2, b_2 \in C$ with $a_1 < b_1$ and $a_2 < b_2$, we have $f(a_1, b_1) = f(a_2, b_2) \leftrightarrow b_1 = b_2$.

(4) For all $a_1, b_1, a_2, b_2 \in C$ with $a_1 < b_1$ and $a_2 < b_2$, we have $f(a_1, b_1) = f(a_2, b_2) \leftrightarrow a_1 = a_2$ and $b_1 = b_2$.

In the general case of an $f\colon [\omega]^n \to \omega$, we get 2^n different possibilities.

Definition 2.3.1. Suppose that $n \geq 1$, $B \subseteq \omega$ is infinite, $f\colon [B]^n \to \omega$, and $u \subseteq n$. We say that a set $C \subseteq B$ is *u-canonical* for f if C is infinite and for all $x_1, x_2 \in [C]^n$, we have $f(x_1) = f(x_2) \leftrightarrow x_1 \upharpoonright u = x_2 \upharpoonright u$. We say that a set $C \subseteq B$ is *canonical* for f if there exists $u \subseteq n$ such that C is u-canonical for f.

Theorem 2.3.2 (Canonical Ramsey Theorem [4]). *Suppose that $n \geq 1$, $B \subseteq \omega$ is infinite, and $f\colon [B]^n \to \omega$. There exists $C \subseteq B$ canonical for f.*

Ramsey's Theorem is an immediate consequence of The Canonical Ramsey Theorem.

Claim 2.3.3. *Suppose that $n, p \geq 1$, $B \subseteq \omega$ is infinite, and $f\colon [B]^n \to p$. If $C \subseteq B$ is canonical for f, then C is homogeneous for f.*

Proof. Suppose that $C \subseteq B$ is u-canonical for f, where $u \subseteq n$. Suppose that there exists $i < n$ such that $i \in u$. Fix $x_k \in [C]^n$ for all $k \in \omega$ such that $x_0 < x_1 < x_2 < \dots$. For any $j, k \in \omega$ with $j \neq k$, we have $x_j \upharpoonright u \neq x_k \upharpoonright u$, hence $f(x_j) \neq f(x_k)$. This contradicts the fact that $f(x_k) < p$ for each $k \in \omega$. It follows that there is no $i < n$ such that $i \in u$, so $u = \emptyset$. Therefore, C is homogeneous for f. $\square$

In the original inductive proof of the Canonical Ramsey Theorem (see [4]), in order to prove the result for exponent $n \geq 2$, Erdös and Rado used Ramsey's Theorem for exponent $2n$ together with the Canonical Ramsey Theorem for exponent $n-1$. Using Theorem 2.2.22, an effective analysis of their proof gives the result that every computable $f\colon [\omega]^2 \to \omega$ has a Π^0_4 canonical set. However, as n increases, the use of induction causes the arithmetical bounds to grow on the order of n^2. Rado [22] discovered a noninductive proof of the Canonical Ramsey Theorem which still used Ramsey's Theorem for exponent $2n$ to prove the result for exponent n. An effective analysis of his proof shows that given $n \geq 2$ and a computable $f\colon [\omega]^n \to \omega$, there exists a Δ^0_{2n+1} canonical set for f.

Below, we give a new, purely inductive proof of the Canonical Ramsey Theorem. Analyzing this proof, we decrease the bounds:

Theorem 2.3.4. *Suppose that $n \geq 2$ and $f\colon [\omega]^n \to \omega$ is computable. There exists a Π^0_{2n-2} set C canonical for f.*

We also analyze the Turing degrees of canonical sets for computable colorings, and show a close connection with the analysis of König's Lemma and Ramsey's Theorem.

Theorem 2.3.5. *Suppose that $n \geq 2$, $\mathbf{a} \gg \mathbf{0}^{(2n-3)}$ and $f\colon [\omega]^n \to \omega$ is computable. There exists $C \subseteq \omega$ such that C is canonical for f and $deg(C) \leq \mathbf{a}$.*

For $n = 2$, we show that the above bounds are sharp.

2.4 The Regressive Function Theorem

As mentioned in Chapter 1, Paris and Harrington [21] discovered a partition theorem slightly stronger than the finite version of Ramsey's Theorem, and Kanamori and McAloon [17] provided another finitary partition theorem, which is true but not provable in Peano Arithmetic. In Chapter 4, we analyze the infinitary version of Kanamori and McAloon's partition theorem from the viewpoint of computability theory.

Definition 2.4.1. Suppose that $n \geq 1$ and $B \subseteq \omega$ is infinite. A function $f\colon [B]^n \to \omega$ is *regressive* if for all $x \in [B]^n$, we have $f(x) < \min(x)$ whenever $\min(x) > 0$, and $f(x) = 0$ whenever $\min(x) = 0$.

Definition 2.4.2. Suppose that $n \geq 1$, $B \subseteq \omega$ is infinite, and $f\colon [B]^n \to \omega$ is regressive. A set $M \subseteq B$ is *minhomogeneous* for f if M is infinite and for all $x, y \in [M]^n$ with $\min(x) = \min(y)$ we have $f(x) = f(y)$.

Theorem 2.4.3 (Regressive Function Theorem [17]). *Suppose that $n \geq 1$, $B \subseteq \omega$ is infinite, and $f\colon [B]^n \to \omega$ is regressive. There exists a minhomogeneous set for f.*

The following claim shows that the Regressive Function Theorem is an immediate consequence of the Canonical Ramsey Theorem.

Claim 2.4.4. *Suppose that $n \geq 1$, $B \subseteq \omega$ is infinite, and $f\colon [B]^n \to \omega$ is regressive. If $C \subseteq B$ is canonical for f, then C is minhomogeneous for f.*

Proof. If $n = 1$, then every infinite subset of ω is minhomogeneous for f, so we may assume that $n \geq 2$. Suppose that $C \subseteq B$ is u-canonical for f, where $u \subseteq n$. Suppose that there exists i with $0 < i < n$ such that $i \in u$. Let $c_0 = \min(C)$. Fix $x_k \in [C]^{n-1}$ for all $k \in \omega$ such that $c_0 < x_0 < x_1 < x_2 < \dots$. For any $j, k \in \omega$ with $j \neq k$, we have $(c_0, x_j) \restriction u \neq (c_0, x_k) \restriction u$, hence $f(c_0, x_j) \neq f(c_0, x_k)$. This contradicts the fact that $f(c_0, x_k) \leq c_0$ for each $k \in \omega$. It follows that there is no i with $0 < i < n$ such that $i \in u$, so either $u = \emptyset$ or $u = \{0\}$. If $u = \emptyset$, then C is homogeneous for f, and hence minhomogeneous for f. If $u = \{0\}$, then for all $x, y \in [C]^n$, we have $f(x) = f(y) \leftrightarrow x \restriction \{0\} = y \restriction \{0\} \leftrightarrow \min(x) = \min(y)$, so C is minhomogeneous for f. $\square$

In Chapter 4, we analyze the Regressive Function Theorem, and establish a sharp characterization for the location of minhomogeneous sets for computable f in terms of the Turing degrees and the arithmetical hierarchy.

Theorem 2.4.5. *Suppose that $n \geq 2$, $f\colon [\omega]^n \to \omega$ is computable and regressive, and $\mathbf{a} \gg \mathbf{0}^{(n-1)}$. There exists a set M minhomogeneous for f such that deg$(M) \leq \mathbf{a}$. Furthermore, for every $n \geq 2$, there exists a computable regressive $f\colon [\omega]^n \to \omega$ such that deg$(M) \gg \mathbf{0}^{(n-1)}$ for every set M minhomogeneous for f.*

Theorem 2.4.6. *Suppose that $n \geq 2$ and $f\colon [\omega]^n \to \omega$ is computable and regressive. There exists a Π^0_n minhomogeneous set for f. Furthermore, for every $n \geq 2$, there exists a computable regressive $f\colon [\omega]^n \to \omega$ with no Σ^0_n set minhomogeneous for f.*

2.5 Summary

Putting together the characterizations of Turing degrees of solutions for computable instances of König's Lemma and the above partition theorems for exponent 2, we see a close connection.

Summary 2.5.1. *Let $\mathbf{a}$ be a Turing degree. The following are equivalent:*

(1) $\mathbf{a} \gg \mathbf{0}'$

(2) Every computable $f\colon [\omega]^2 \to 2$ has a homogeneous set H such that $deg(H)' \leq \mathbf{a}$.

(3) Every computable regressive $f\colon [\omega]^2 \to \omega$ has a minhomogeneous set M such that $deg(M) \leq \mathbf{a}$.

(4) Every computable $f\colon [\omega]^2 \to \omega$ has a canonical set C such that $deg(C) \leq \mathbf{a}$.

For exponents $n \geq 3$, the Turing degrees characterizing the location of solutions for Ramsey's Theorem and the Regressive Function Theorem increase by one jump for each successive value of n, while our upper bounds for solutions for the Canonical Ramsey Theorem increase by two jumps for each successive value of n.

In terms of the arithmetical hierarchy, each of the above partition theorems for exponent 2 have Π^0_2 solutions for computable instances, but not necessarily Σ^0_2 solutions. For exponents $n \geq 3$ the location of solutions for Ramsey's Theorem and the Regressive Function Theorem increase by one jump for each successive value of n, while our upper bounds for solutions for the Canonical Ramsey Theorem increase by two jumps for each successive value of n.

Chapter 3

The Canonical Ramsey Theorem and Computability Theory

3.1 A New Proof

Our proof of the Canonical Ramsey Theorem is inductive and similar in broad outline to the proof of Ramsey's Theorem given in Chapter 2. The basic question is how to define a "precanonical pair" (A, g) so that that we can carry out the same outline to prove the Canonical Ramsey Theorem. For simplicity, consider a function $f\colon [\omega]^2 \to \omega$. We will enumerate A in increasing order as $a_0, a_1, \dots$. We begin by letting $a_0 = 0$. If there exists $c \in \omega$ such that there are infinitely many $b \in \omega$ with $f(a_0, b) = c$, then we can define $g(a_0) = c$, restrict attention to the set $I_0 = \{b \in \omega : f(a_0, b) = c\}$, and after letting $a_1 = \min I_0$, continue in this fashion. In this case, we've made progress toward achieving a u-canonical set with $1 \notin u$, because if we fix a_0 and vary $b \in I_0$, we do not change the value of f. If we succeed infinitely many times in this manner with a fixed c, then the corresponding elements form a $\emptyset$-canonical set, while if we succeed with infinitely many different c in this manner, then the corresponding elements form a $\{0\}$-canonical set. Notice that this decision (one fixed c versus infinitely many distinct c) amounts to finding a canonical set for exponent 1 for g restricted to the set of successes.

The problem arises when for each $c \in \omega$, there are only finitely many $b \in \omega$ with $f(a_0, b) = c$. Now we must seek to make progress toward achieving a u-canonical set with $1 \in u$. We therefore let $I_0 = \{b \in \omega : f(a_0, b) \neq f(a_0, b')$ for all $b' < b\}$, so that if we fix a_0 and vary $b \in I_0$, we always change the value of f. We now want to let $g(a_0)$ be some new, infinitary color d distinct from each $c \in \omega$. Suppose that we then set $a_1 = \min I_0$, and again are faced with the situation that for each $c \in \omega$, there are only finitely many $b \in I_0$ with $f(a_1, b) = c$. We first want to thin out I_0 to an infinite set I_0' so that $f(a_i, b_0) = f(a_j, b_1) \to b_0 = b_1$ whenever $0 \leq i, j \leq 1$ and $b_0, b_1 \in I_0'$ (which is possible by the assumption about a_0, a_1). This allows both a_0 and a_1 to be in the same u-canonical set with $1 \in u$. Next, we need to assign an appropriate infinitary color to $g(a_1)$ so that a canonical set for g will be an u-canonical set for f. Thus, if the set $\{b \in I_0' : f(a_0, b) = f(a_1, b)\}$ is infinite, we let $g(a_1) = g(a_0)$ and we let I_1 be this set. Otherwise we will set $g(a_1)$ to a new infinitary color and let $I_1 = \{b \in I_0' : f(a_0, b) \neq f(a_1, b)\}$. If we succeed infinitely many times in this manner with a fixed infinitary color d, then the corresponding elements form a $\{1\}$-canonical set, while if we succeed with infinitely many different d in this manner, then the corresponding elements form a $\{0, 1\}$-canonical set. Notice again that this decision (one fixed d versus infinitely many distinct d) amounts to finding a canonical set for exponent 1 for g restricted to those elements assigned infinitary colors.

In general, given $f\colon [B]^{n+1} \to \omega$, we can pursue the above strategy to get an infinite set $A \subseteq B$ and a function $g\colon [A]^n \to \omega \times 2$, where we interpret each $(c, 0) \in \omega \times 2$ as a finitary color and each $(d, 1) \in \omega \times 2$ as an infinitary color. Now, before we can apply induction, it is important to thin out our set A to a set D so that either g maps all elements of $[D]^n$ to finitary colors, or g maps all elements of $[D]^n$ to infinitary colors. Of course, we can do this with a simple application of Ramsey's Theorem for exponent n. Although this strategy will succeed in proving the Canonical Ramsey Theorem, the use of Ramsey's Theorem is costly to an effective analysis. We therefore pursue a slightly different approach which will roll this use of Ramsey's Theorem into the induction. Hence, we extend the notion of canonical sets to functions $f\colon [B]^n \to \omega \times p$ for $p \in \omega$ by also stipulating that a canonical C set must have the property that f maps all elements of $[C]^n$ into the same column of $\omega \times p$. Then, the above strategy will give us an infinite set A and a function $g\colon [A]^n \to \omega \times 2p$, where we interpret $(c, q) \in \omega \times 2p$ with $0 \leq q < p$ as a finitary color corresponding to column q and $(d, q) \in \omega \times 2p$ with $p \leq q < 2p$ as an infinitary color corresponding to column $q - p$ of $\omega \times p$. Applying induction to this g will give us the result because the resulting canonical set will be mapped by g entirely into one column of $\omega \times 2p$.

Definition 3.1.1. Suppose that $n, p \geq 1$, $B \subseteq \omega$ is infinite, $f\colon [B]^n \to \omega \times p$, and $u \subseteq n$. We say that a set C is *u-canonical* for f if

(1) $C \subseteq B$

(2) C is infinite.

(3) C is homogeneous for $\pi_2 \circ f\colon [B]^n \to p$.

(4) If $x_1, x_2 \in [C]^n$, then $f(x_1) = f(x_2) \leftrightarrow x_1 \restriction u = x_2 \restriction u$.

We say that a set C is *canonical* for f if there exists $u \subseteq n$ such that C is u-canonical for f.

Remark 3.1.2. Suppose that $n \geq 1$, $B \subseteq \omega$ is infinite, $f \colon [B]^n \to \omega$, and $u \subseteq n$. Define $f^* \colon [B]^n \to \omega \times 1$ by letting $f^*(x) = (f(x), 0)$. Notice that for any infinite set $C \subseteq \omega$, C is u-canonical for f (as in Definition 2.3.1) if and only if C is u-canonical for f^* (as in Definition 3.1.1). Therefore, in the following, we identify a function $f \colon [B]^n \to \omega$ with the corresponding function $f^* \colon [B]^n \to \omega \times 1$.

For the reasons mentioned above, we prove the Canonical Ramsey Theorem by induction on n in the following strong form.

Theorem 3.1.3. *Suppose that $n, p \geq 1$, $B \subseteq \omega$ is infinite, and $f \colon [B]^n \to \omega \times p$. There exists a set $C \subseteq B$ such that C is canonical for f.*

Definition 3.1.4. Suppose that $n, p \geq 1$, $B \subseteq \omega$ is infinite, and $f \colon [B]^{n+1} \to \omega \times p$. We call a pair (A, g), where $A \subseteq B$ is infinite and $g \colon [A]^n \to \omega \times 2p$, a *precanonical pair* for f if:

(1) For all $x \in [A]^n$ with $g(x) = (c, q)$ where $0 \leq q < p$, we have $f(x, a) = (c, q)$ for all $a \in A$ with $a > x$.

(2) For all $x \in [A]^n$ with $g(x) = (d, q)$ where $p \leq q < 2p$, we have $\pi_2(f(x, a)) = q - p$ for all $a \in A$ with $a > x$.

(3) For all $x_1, x_2 \in [A]^n$ with $g(x_1) = (d_1, q)$ and $g(x_2) = (d_2, q)$ where $p \leq q < 2p$, and all $a_1, a_2 \in A$ with $a_1 > x_1$ and $a_2 > x_2$,

   (a) If $a_1 \neq a_2$, then $f(x_1, a_1) \neq f(x_2, a_2)$

   (b) If $a_1 = a_2$, then $f(x_1, a_1) = f(x_2, a_2) \leftrightarrow d_1 = d_2$.

We first show that the above definition of "precanonical pair" allows our outline to succeed.

Claim 3.1.5. *Suppose that $n, p \geq 1$, $B \subseteq \omega$ is infinite, $f \colon [B]^{n+1} \to \omega \times p$, and (A, g) is a precanonical pair for f. Suppose that $C \subseteq A$ is u-canonical for g, where $u \subseteq n$.*

(1) If $\pi_2(g([C]^n)) = \{q\}$ where $0 \leq q < p$, then C is u-canonical for f (now viewing u as a subset of $n+1$).

(2) If $\pi_2(g([C]^n)) = \{q\}$ where $p \leq q < 2p$, then C is $(u \cup \{n\})$-canonical for f.

Proof. (1) For any $x \in [C]^n$ and $a \in C$ with $x < a$, we have $\pi_2(f(x, a)) = \pi_2(g(x)) = q$ by condition (1) of Definition 3.1.4, hence C is homogeneous for $\pi_2 \circ f$. Let $x_1, x_2 \in [C]^n$, $a_1, a_2 \in C$ with $x_1 < a_1$ and $x_2 < a_2$. By condition (1) of Definition 3.1.4, we have $f(x_1, a_1) = g(x_1)$ and $f(x_2, a_2) = g(x_2)$. Therefore, $f(x_1, a_1) = f(x_2, a_2) \leftrightarrow g(x_1) = g(x_2) \leftrightarrow x_1 \restriction u = x_2 \restriction u$. Hence, C is u-canonical for f.

(2) For any $x \in [C]^n$ and $a \in C$ with $x < a$, we have $\pi_2(f(x, a)) = q - p$ by condition (2) of Definition 3.1.4, hence C is homogeneous for $\pi_2 \circ f$. Let $x_1, x_2 \in [C]^n$, $a_1, a_2 \in C$ with $x_1 < a_1$ and $x_2 < a_2$. Suppose

first that $x_1 \restriction u = x_2 \restriction u$ and $a_1 = a_2$. Then $g(x_1) = g(x_2)$ and $a_1 = a_2$. Therefore, by condition (3b) of Definition 3.1.4, we have $f(x_1, a_1) = f(x_2, a_2)$. Suppose now that $x_1 \restriction u \neq x_2 \restriction u$ or $a_1 \neq a_2$. If $a_1 \neq a_2$, then $f(x_1, a_1) \neq f(x_2, a_2)$ by condition (3a) of Definition 3.1.4. If $x_1 \restriction u \neq x_2 \restriction u$ and $a_1 = a_2$, then $g(x_1) \neq g(x_2)$ and $a_1 = a_2$, so $f(x_1, a_1) \neq f(x_2, a_2)$ by condition (3b) of Definition 3.1.4. Therefore, $f(x_1, a_1) = f(x_2, a_2)$ if and only if $x_1 \restriction u = x_2 \restriction u$ and $a_1 = a_2$, so C is $(u \cup \{n\})$-canonical for f. ☐

Next, we show that precanonical pairs exist by a method along the lines of Proof 1 of Proposition 2.2.5. We build a precanonical pair (A, g) in stages which consist of selecting a new element for A and thinning out the set of potential later elements to make them acceptable to the new element and its chosen color. To facilitate this construction, we first define a notion of precanonical triple which will provide an approximation to a desired precanonical pair.

Definition 3.1.6. Suppose that $n, p \geq 1$, $B \subseteq \omega$ is infinite, and $f \colon [B]^{n+1} \to \omega \times p$. We call a triple (z, I, g) where $z \subseteq B$ is finite, $I \subseteq B$ is infinite, $z < I$, and $g \colon [z]^n \to \omega \times 2p$, a *precanonical triple* for f if:

(1) For all $x \in [z]^n$ with $g(x) = (c, q)$ where $0 \leq q < p$, we have $f(x, a) = (c, q)$ for all $a \in z \cup I$ with $a > x$.

(2) For all $x \in [z]^n$ with $g(x) = (d, q)$ where $p \leq q < 2p$, we have $\pi_2(f(x, a)) = q - p$ for all $a \in z \cup I$ with $a > x$.

(3) For all $x_1, x_2 \in [z]^n$ with $g(x_1) = (d_1, q)$ and $g(x_2) = (d_2, q)$ where $p \leq q < 2p$, and all $a_1, a_2 \in z \cup I$ with $a_1 > x_1$ and $a_2 > x_2$,

 (a) If $a_1 \neq a_2$, then $f(x_1, a_1) \neq f(x_2, a_2)$

 (b) If $a_1 = a_2$, then $f(x_1, a_1) = f(x_2, a_2) \leftrightarrow d_1 = d_2$.

Proposition 3.1.7. *Suppose that $n, p \geq 1$, $B \subseteq \omega$ is infinite, and $f \colon [B]^{n+1} \to \omega \times p$. There exists a precanonical pair (A, g) for f.*

Proof. We inductively define a sequence $(a_m, I_m, g_m)_{m \in \omega}$ such that

- $a_m \in B$.
- $I_m \subseteq B$ is infinite.
- $g_m \colon [\{a_i : i \leq m\}]^n \to \omega \times 2p$.
- $a_0 < a_1 < \cdots < a_{m-1} < a_m < I_m \subseteq I_{m-1} \subseteq \cdots \subseteq I_1 \subseteq I_0$.
- $g_0 \subseteq g_1 \subseteq \cdots \subseteq g_m$.

- $(\{a_i : i \leq m\}, I_m, g_m)$ is a precanonical triple for f.

We begin by letting $a_{-1} = -1$, $I_{-1} = B$, and $g_{-1} = \emptyset$. Suppose that we have defined our sequence through stage $m \geq -1$. We first let $a_{m+1} = \min(I_m)$, $I'_m = I_m - \{a_{m+1}\}$, and $g_{m+1}(x) = g_m(x)$ for all $x \in [\{a_i : i \leq m\}]^n$. Let $\mathcal{F} = [\{a_i : i \leq m+1\}]^n$, and list the elements of $\mathcal{F}$ whose greatest element is a_{m+1} as $x_0 <_n x_1 <_n \cdots <_n x_{l-1}$ (recall Definition 1.2.2), where $l = \binom{m+1}{n-1}$. If $l = 0$, let $I_{m+1} = I'_m$ and $g_{m+1} = g_m$. Otherwise, we proceed inductively through the x_k, defining $g_{m+1}(x_k)$ and infinite sets $J_0, J_1, \ldots, J_l$ such that $I'_m = J_0 \supseteq J_1 \supseteq \cdots \supseteq J_l$ along the way. Let $J_0 = I'_m$. Suppose that $k < l$, and we have defined J_k. First, since J_k is infinite, there exists $q < p$ such that there are infinitely many $b \in J_k$ with $\pi_2(f(x_k, b)) = q$. Fix the least such q, and let $H_k = \{b \in J_k : \pi_2(f(x_k, b)) = q\}$.

Case 1: *There exists $c \in \omega$ such that there are infinitely many $b \in H_k$ with $f(x_k, b) = (c, q)$.* In this case, let $g_{m+1}(x_k) = (c, q)$ and let $J_{k+1} = \{b \in H_k : f(x_k, b) = (c, q)\}$. Proceed to the next value of $k \leq l$, if it exists.

Case 2: *Otherwise, for every $c \in \omega$, there are only finitely many $b \in H_k$ with $f(x_k, b) = (c, q)$.* Let $\mathcal{D} = \{y \in \mathcal{F} : y <_n x_k \text{ and } \pi_2(g_{m+1}(y)) = q + p\}$, and notice that for each $y \in \mathcal{D}$ and each $c \in \omega$, there is at most one $b \in H_k$ with $f(y, b) = (c, q)$ (if $\max(y) < a_{m+1}$, this follows from the fact that $(\{a_i : i \leq m\}, I_m, g_m)$ is a precanonical triple for f and $H_k \subseteq I_m$, while if $\max(y) = a_{m+1}$, say $y = x_i$ with $1 \leq i < k$, this follows from the fact that $H_k \subseteq J_{i+1}$, so $f(y, b_1) \neq f(y, b_2)$ for all $b_1, b_2 \in H_k$ with $b_1 \neq b_2$ by construction). We now inductively define an increasing $h \colon \omega \to H_k$ such that $f(x_k, h(i)) \neq f(y, h(j))$ whenever $i \neq j \in \omega$ and $y \in \mathcal{D} \cup \{x_k\}$. Let $h(0) = \min(H_k)$. Suppose that we have defined $h(t)$. By the assumption of Case 2 and the above comments, there exists $b \in H_k$ with $b > h(t)$ such that $f(x_k, b) \notin \{f(y, h(i)) : y \in \mathcal{D} \cup \{x_k\}, 0 \leq i \leq t\}$ and $f(y, b) \notin \{f(x_k, h(i)) : 0 \leq i \leq t\}$ for all $y \in \mathcal{D}$ (since each of these sets is finite), and we let $h(t+1)$ be the least such b. Let $H'_k = \{h(t) : t \in \omega\}$.

Subcase 1: *There exists $y \in \mathcal{D}$ such that $\{b \in H'_k : f(x_k, b) = f(y, b)\}$ is infinite.* In this case, choose the least such y (under the ordering $<_n$), let $g_{m+1}(x_k) = g_{m+1}(y)$, and let $J_{k+1} = \{b \in H'_k : f(x_k, b) = f(y, b)\}$. Proceed to the next value of $k < l$, if it exists.

Subcase 2: *Otherwise, for every $y \in \mathcal{D}$, there are only finitely many $b \in H'_k$ with $f(x_k, b) = f(y, b)$.* Thus, there are only finitely many $b \in H'_k$ such that there exists $y \in \mathcal{D}$ with $f(x_k, b) = f(y, b)$. Let $g_{m+1}(x_k) = (d, q + p)$, where d is least such that $g_{m+1}(y) \neq (d, q + p)$ for all $y \in \mathcal{D}$ and let $J_{k+1} = \{b \in H'_k : f(x_k, b) \neq f(y, b) \text{ for all } y \in \mathcal{D}\}$. Proceed to the next value of $k < l$, if it exists.

Once we reach $k = l$, we let $I_{m+1} = J_l$. One easily checks that the invariants are maintained (i.e. that $a_m < a_{m+1} < I_{m+1} \subseteq I_m$, $g_m \subseteq g_{m+1}$, and $(\{a_i : i \leq m + 1\}, I_{m+1}, g_{m+1})$ is a precanonical triple for f). This completes stage $m + 1$.

Finally, let $A = \{a_m : m \in \omega\}$ and $g = \bigcup_{m\in\omega} g_m$. Then (A, g) is a precanonical pair for f. $\square$

Proof of Theorem 3.1.3. The proof is by induction on n. Suppose that $n = 1$ so that we have $f\colon B \to \omega \times p$. Fix an infinite $A \subseteq B$ and $q < p$ such that $\pi_2(f(A)) = \{q\}$. If there exists a $c \in \omega$ such that there are infinitely many $a \in A$ with $f(a) = (c, q)$, let $C = \{a \in A : f(a) = (c, q)\}$, and notice that $f(a_1) = f(a_2)$ for all $a_1, a_2 \in C$, so C is $\emptyset$-canonical for f. Otherwise, there are infinitely many $c \in \omega$ such that there is an $a \in A$ with $f(a) = (c, q)$. Letting $C = \{a \in A : f(a) \neq f(b)$ for all $b < a$ with $b \in A\}$, we see that $f(a_1) \neq f(a_2)$ for all $a_1, a_2 \in C$, so C is $\{0\}$-canonical for f.

Suppose that the theorem holds for n, and we're given $f\colon [B]^{n+1} \to \omega \times p$. By Proposition 3.1.7, there exists a precanonical pair (A, g) for f. Applying the inductive hypothesis to $g\colon [A]^n \to \omega \times 2p$, there exists $C \subseteq A$ which is canonical for g. By Claim 3.1.5, C is canonical for f. $\square$

3.2 Computability-Theoretic Analysis

If we analyze the proof of Proposition 3.1.7 for a given computable B and computable $f\colon [B]^n \to \omega \times p$, we can easily see that there exists a precanonical pair (A, g) for f with $A \oplus g \leq_T 0'''$. It seems that we need a $0'''$-oracle to decide the 3-quantifier ($\exists\forall\exists$) question of whether to enter Case 1 or Case 2. However, by making use of an r-cohesive set, we can lower the complexity to a 2-quantifier question.

Recall the characterization of the Turing degrees of jumps of r-cohesive sets from Theorem 2.2.10. The Low Basis Theorem relative to $\mathbf{0}'$ yields an $\mathbf{a} \gg \mathbf{0}'$ such that $\mathbf{a}' = \mathbf{0}''$. Using this $\mathbf{a}$ in Theorem 2.2.10 yields the following corollary.

Corollary 3.2.1 (Jockusch and Stephan [9]). *There exists an r-cohesive set V such that $V'' \leq_T 0''$.*

Below, we will need r-cohesive sets of low complexity inside a given infinite computable set. The following easy lemma provides these.

Lemma 3.2.2. *Suppose that B is an infinite computable set. If V is r-cohesive, then $p_B(V) \subseteq B$ is r-cohesive and $p_B(V) \equiv_T V$.*

Proof. Notice that $p_B(V)$ is infinite and $p_B(V) \equiv_T V$ because p_B is computable and strictly increasing. Let Z be a computable set. Since V is r-cohesive and $p_B^{-1}(Z)$ is computable, either $V \cap p_B^{-1}(Z)$ is finite or $V \cap \overline{p_B^{-1}(Z)}$ is finite. If $V \cap p_B^{-1}(Z)$ is finite, then $V \subseteq^* \overline{p_B^{-1}(Z)}$, so $p_B(V) \subseteq^* p_B(\overline{p_B^{-1}(Z)}) \subseteq \overline{Z}$, and hence $p_B(V) \cap Z$ is finite. If $V \cap \overline{p_B^{-1}(Z)}$ is finite, then $V \subseteq^* p_B^{-1}(Z)$, so $p_B(V) \subseteq^* p_B(p_B^{-1}(Z)) \subseteq Z$, and hence $p_B(V) \cap \overline{Z}$ is finite. It follows that $p_B(V)$ is r-cohesive. $\square$

Proposition 3.2.3. *Suppose that $n, p \geq 1$, $B \subseteq \omega$ is infinite and computable, and $f\colon [B]^{n+1} \to \omega \times p$ is computable. There exists a precanonical pair (A, g) for f such that $A \oplus g \leq_T 0''$. Furthermore, if $n = 1$ and $\mathbf{a} \gg \mathbf{0}'$, then there exists a precanonical pair (A, g) for f with deg$(A \oplus g) \leq \mathbf{a}$.*

Proof. By Corollary 3.2.1 and Lemma 3.2.2, there exists an r-cohesive set $V \subseteq B$ such that $V'' \leq_T 0''$. For each $x \in [V]^n$ and each $(c, q) \in \omega \times p$, the set $Z_{(c,q)} = \{b \in B : f(x, b) = (c, q)\}$ is computable, so either $V \cap Z_{(c,q)}$ is finite or $V \cap \overline{Z_{(c,q)}}$ is finite.

We now carry out the above existence proof of a precanonical pair for $f \restriction [V]^n\colon [V]^n \to \omega \times p$ using a V''-oracle and characteristic indices (relative to V) for all infinite sets. As we proceed through the proof, the first noncomputable (relative to V) step is the construction of H_k, where we need to find the least $q < p$ such that $H_k = \{b \in J_k : \pi_2(f(x_k, b)) = q\}$ is infinite, which we can do using a V''-oracle. Next, we need to decide whether to enter Case 1 or Case 2. By the last sentence of the above paragraph, we enter Case 1 if and only if $(\exists c)(\exists m)(\forall b)[b \in H_k \wedge b \geq m \to f(x_k, b) = (c, q)]$. Again, we can decide this question using a V''-oracle. If we enter Case 2, the next noncomputable (relative to V) step is the decision whether to enter Subcase 1 or Subcase 2. Since $\mathcal{D}$ is finite, and for each $y \in \mathcal{D}$ we need to determine whether a given V-computable set is infinite, we can again decide this question using a V''-oracle. The rest of the steps of the proof are V-computable, so we end up with a precanonical pair (A, g) for $f \restriction [V]^n$ (hence for f) such that $A \oplus g \leq_T V'' \leq_T 0''$.

Suppose now that $n = 1$ and $\mathbf{a} \gg \mathbf{0}'$. By Theorem 2.2.10 and Lemma 3.2.2, there exists an r-cohesive set $V \subseteq B$ such that $\deg(V)' \leq \mathbf{a}$. For each $a \in B$ and $q < p$, the set $Z_q = \{b \in B : \pi_2(f(a, b)) = q\}$ is computable, so either $V \cap Z_q$ is finite or $V \cap \overline{Z_q}$ is finite (since V is r-cohesive). Therefore, for each $a \in V$, $\lim_{b \in V} \pi_2(f(a, b))$ exists, and we denote its value by q_a. Notice that we can use a V'-oracle to compute q_a given $a \in V$. Similarly, for each $a \in V$ and $c \in \omega$, the set $Z_c = \{b \in B : \pi_1(f(a, b)) = c\}$ is computable, so either $V \cap Z_c$ is finite or $V \cap \overline{Z_c}$ is finite. Therefore, for each $a \in V$, either $\lim_{b \in V} \pi_1(f(a, b))$ exists (and is finite) or $\lim_{b \in V} \pi_1(f(a, b)) = \infty$.

Let $Y = \{a \in V : \lim_{b \in V} \pi_1(f(a, b)) < \infty\}$. Notice that $a \in Y$ if and only if $(\exists c)(\exists m)(\forall b)[(b \geq m \wedge b \in V) \to \pi_1(f(a, b)) = c]$, hence $Y \in \Sigma_2^{0,V}$.

Case 1: *Y is infinite:* Choose an infinite $I \subseteq Y$ such that $I \leq_T V'$. For each $a \in V$, we can use a V'-oracle to determine whether $a \in I$, and if so to compute $c_a = \lim_{b \in V} \pi_1(f(a, b))$. We now construct a precanonical pair (A, g) for f using a V'-oracle. First, let a_0 be the least element of I and let $g(a_0) = (c_{a_0}, q_{a_0})$. If we have already defined $a_0, a_1, \ldots, a_m$, let a_{m+1} be the least $b \in I$ such that $b > a_m$ and $f(a_i, b) = g(a_i) = (c_{a_i}, q_{a_i})$ for all i with $0 \leq i \leq m$, and let $g(a_{m+1}) = (c_{a_{m+1}}, q_{a_{m+1}})$. Letting $A = \{a_m : m \in \omega\}$, we see that (A, g) is a precanonical pair for $f \restriction [V]^n$ (hence for f) such that $\deg(A \oplus g) \leq \deg(V)' \leq \mathbf{a}$.

Case 2: *Y is finite:* Fix α such that $\lim_{b\in V} \pi_1(f(a,b)) = \infty$ for all $a \in V$ with $a > \alpha$. We now construct a precanonical pair (A, g) for f using a V'-oracle. First, let a_0 be the least element of V greater than α, and let $g(a_0) = (0, p+q_{a_0})$. Suppose that we have already defined $a_0, a_1, \ldots, a_m$ and $g(a_0), g(a_1), \ldots, g(a_m)$, and assume inductively that for all sufficiently large $b \in V$, we have

(1) For all $i \leq m$, $\pi_2(f(a_i, b)) = q_{a_i}$.

(2) For all $i, j, k \leq m$ with $i < k$ and $q_{a_i} = q_{a_j}$, $f(a_i, a_k) \neq f(a_j, b)$.

(3) For all $i, j \leq m$ with $q_{a_i} = q_{a_j}$, $f(a_i, b) = f(a_j, b) \leftrightarrow g(a_i) = g(a_j)$.

Using a V'-oracle, let a_{m+1} be the least $b \in V$ such that $b > a_m$ and (1), (2), and (3) hold for b. Let $D = \{i \in \omega : 0 \leq i \leq m \text{ and } q_{a_i} = q_{a_{m+1}}\}$. For each $i \in D$, the set $Z_i = \{b \in B : b > a_{m+1}$ and $f(a_i, b) = f(a_{m+1}, b)\}$ is computable, so either $V \cap Z_i$ is finite or $V \cap \overline{Z_i}$ is finite. Also, the set $Z_\infty = \{b \in B : b > a_{m+1}$ and $f(a_{m+1}, b) \notin \{f(a_i, b) : i \in D\}\}$ is computable, so either $V \cap Z_\infty$ is finite or $V \cap \overline{Z_\infty}$ is finite. Putting this together with the fact that the sets in the list $(V \cap Z_i)_{i\in D\cup\{\infty\}}$ are pairwise disjoint and have union equal to $\{b \in V : b > a_{m+1}\}$, it follows that exists exactly one $j \in D \cup \{\infty\}$ with $V \cap \overline{Z_j}$ finite. Moreover, we can find this j using a V'-oracle (by running through $\beta \in B$ in increasing order and asking a V'-oracle if all elements of V greater than β lie in one fixed Z_i). If $j \in D$, let $g(a_{m+1}) = g(a_j)$, and if $j = \infty$, let $g(a_{m+1}) = (d, p + q_{a_{m+1}})$, where d is the least element of $\omega - \{\pi_1(g(a_i)) : i \in D\}$. Then for all sufficiently large $b \in V$, we have

(1) For all $i \leq m+1$, $\pi_2(f(a_i, b)) = q_{a_i}$.

(2) For all $i, j, k \leq m+1$ with $i < k$ and $q_{a_i} = q_{a_j}$, $f(a_i, a_k) \neq f(a_j, b)$.

(3) For all $i, j \leq m+1$ with $q_{a_i} = q_{a_j}$, $f(a_i, b) = f(a_j, b) \leftrightarrow g(a_i) = g(a_j)$.

Hence, the induction hypothesis holds, and we may continue. Letting $A = \{a_m : m \in \omega\}$, we see that (A, g) is a precanonical pair for $f \upharpoonright [V]^n$ (hence for f) such that $\deg(A \oplus g) \leq \deg(V)' \leq \mathbf{a}$. $\square$

We are now in a position to give upper bounds on the Turing degrees of canonical sets for computable f. We prove the result in relativized form to facilitate the induction.

Theorem 3.2.4. *Suppose that $X \subseteq \omega$, $n, p \geq 1$, $B \subseteq \omega$ is infinite and X-computable, and $f\colon [B]^n \to \omega \times p$ is X-computable. If $n = 1$, then there exists an X-computable set $C \subseteq B$ canonical for f. If $n \geq 2$ and $\mathbf{a} \gg deg(X)^{(2n-3)}$, there exists a set $C \subseteq B$ canonical for f such that $deg(C) \leq \mathbf{a}$.*

Proof. We prove the theorem by induction on n. First, if $n = 1$, notice that the set C produced in the base case of the proof of Theorem 3.1.3 is X-computable if both B and f are. Suppose now that $n = 2$, B and $f\colon [B]^2 \to \omega \times p$ are X-computable, and $\mathbf{a} \gg \deg(X)'$. By Proposition 3.2.3 relativized to X, there exists a precanonical pair (A, g) for f with $\deg(A \oplus g) \leq \mathbf{a}$. By the inductive hypothesis, there exists a set C canonical for $g\colon [A]^1 \to \omega \times 2p$ with $\deg(C) \leq \deg(A \oplus g) \leq \mathbf{a}$. By Claim 3.1.5, C is canonical for f.

Suppose that $n \geq 2$ and the theorem holds for n. Suppose that B and $f\colon [B]^{n+1} \to \omega \times p$ are X-computable, and $\mathbf{a} \gg \deg(X)^{(2n-1)}$. By Proposition 3.2.3 relativized to X, there exists a precanonical pair (A, g) for f with $A \oplus g \leq X''$. Applying the inductive hypothesis to $g\colon [A]^n \to \omega \times 2p$, there exists $C \subseteq A$ canonical for $g\colon [A]^n \to \omega \times 2p$ with $\deg(C) \leq \mathbf{a}$ since $\mathbf{a} \gg \deg(X)^{(2n-1)} = (\deg(X)'')^{(2n-3)} \geq \deg(A \oplus g)^{(2n-3)}$. By Claim 3.1.5, C is canonical for f. □

We immediately obtain bounds for the location of canonical sets in the arithmetical hierarchy. These bounds will be improved in the next section.

Corollary 3.2.5. *Suppose that $n \geq 2$, $p \geq 1$, $B \subseteq \omega$ is infinite and computable, and $f\colon [B]^n \to \omega \times p$ is computable. There exists a Δ^0_{2n-1} set $C \subseteq B$ canonical for f.*

Proof. As noted in the Chapter 2, a relativization of Theorem 2.1.4 implies that $\mathbf{0}^{(2n-2)} \gg \mathbf{0}^{(2n-3)}$. Therefore, by Theorem 3.2.4, there exists a set $C \subseteq B$ canonical for f such that $\deg(C) \leq \mathbf{0}^{(2n-2)}$. Any such C is Δ^0_{2n-1}. □

The proof of Proposition 3.2.3 for the case $n = 1$ relied on the ability to form a set of reasonably low complexity which either consisted entirely of elements needing to be assigned finitary colors, or entirely of elements needing to be assigned infinitary colors. We next show that this special feature of $n = 1$ is essential to finding precanonical pairs below any $\mathbf{a} \gg \mathbf{0}'$.

Theorem 3.2.6. *There exists a computable $f\colon [\omega]^3 \to \omega$ such that $\deg(A) \geq \mathbf{0}''$ whenever (A, g) is a precanonical pair for f.*

Proof. By Proposition 2.2.19, there exists a c.e. $h_0\colon [\omega]^2 \to 2$ (that is, $\{x \in [\omega]^2 : h_0(x) = 1\}$ is c.e.) such that for all sets H homogeneous for h_0, we have $h_0([H]^2) = \{0\}$ and $H \geq_T 0'$. By the same result relative to $0'$, there exists a $0'$-c.e. $h_1\colon [\omega]^2 \to 2$ (that is, $\{x \in [\omega]^2 : h_1(x) = 1\}$ is $0'$-c.e.) such that for all sets H homogeneous for h_1, we have $h_1([H]^2) = \{0\}$ and $H \oplus 0' \geq_T 0''$.

Define $h\colon [\omega]^2 \to 2$ by

$$h(x) = \begin{cases} 1 & \text{if either } h_0(x) = 1 \text{ or } h_1(x) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Notice that $\{x \in [\omega]^2 : h(x) = 1\}$ is $0'$-c.e. Suppose that H is homogeneous for h. Then $h([H]^2) = \{0\}$ because if $h([H]^2) = \{1\}$, then an application of Ramsey's Theorem to the function $h_2 \colon [H]^2 \to 2$ given by

$$h_2(x) = \begin{cases} 0 & \text{if } h_0(x) = 1 \\ 1 & \text{if } h_0(x) = 0 \text{ and } h_1(x) = 1 \end{cases}$$

would give an infinite set I such that either $h_1([I]^2) = \{1\}$ or $h_2([I]^2) = \{1\}$, a contradiction. Thus, H is homogeneous for both h_1 and h_2. It follows that $H \geq_T 0'$ and hence $H \geq_T H \oplus 0' \geq_T 0''$.

Since $\{x \in [\omega]^2 : h(x) = 1\}$ is $0'$-c.e., it is Σ^0_2, so there exists a computable $R(x, a, b)$ such that $h(x) = 1 \leftrightarrow (\exists a)(\forall b)R(x, a, b)$ for all $x \in [\omega]^2$. Define $f \colon [\omega]^3 \to \omega$ as follows. Given $x \in [\omega]^2$ and $s \in \omega$ with $x < s$, let

$$f(x, s) = \begin{cases} (\mu a < s)(\forall b < s)R(x, a, b) & \text{if } (\exists a < s)(\forall b < s)R(x, a, b) \\ s & \text{otherwise} \end{cases}$$

Notice that f is computable. Furthermore, for all $x \in [\omega]^2$, we have $h(x) = 1 \leftrightarrow \lim_s f(x, s)$ exists and is finite, and $h(x) = 0 \leftrightarrow \lim_s f(x, s) = \infty$. Suppose that (A, g) is a precanonical pair for f. For any $y \in [A]^2$, we either have $f(y, a_1) = f(y, a_2)$ for all $a_1, a_2 \in A$ with $y < a_1 < a_2$ (if $\pi_2(g(y)) = 0$), or $f(y, a_1) \neq f(y, a_2)$ for all $a_1, a_2 \in A$ with $y < a_1 < a_2$ (if $\pi_2(g(y)) = 1$). Therefore, given $y \in [A]^2$, if we let $a_1, a_2 \in A$ be least such that $y < a_1 < a_2$, we have $h(y) = 1 \leftrightarrow f(y, a_1) = f(y, a_2)$ and $h(y) = 0 \leftrightarrow f(y, a_1) \neq f(y, a_2)$. Hence, $h \upharpoonright [A]^2 \colon [A]^2 \to 2$ is A-computable. Since every set H homogeneous for $h \upharpoonright [A]^2$ satisfies $h([H]^2) = \{0\}$, it follows from [12, Theorem 5.11] (relativized to A) that $h \upharpoonright [A]^2$, and hence h itself, has an A-computable homogeneous set. Since every set homogeneous for h has degree above $\mathbf{0}''$, we have $\deg(A) \geq \mathbf{0}''$. □

Therefore, the bounds for canonical sets given by Theorem 3.2.4 are the best possible from an effective analysis of the above proof of the Canonical Ramsey Theorem. We show in Chapter 4 that the bound given by Theorem 3.2.4 for exponent 2 is sharp.

3.3 Arithmetical Bounds

Corollary 3.2.5 provided bounds in the arithmetical hierarchy for canonical sets for computable $f \colon [\omega]^n \to \omega \times p$. In particular, we established that every computable $f \colon [\omega]^2 \to \omega$ has a Δ^0_3 canonical set. We first improve this result by showing that every computable $f \colon [\omega]^2 \to \omega \times p$ has a Π^0_2 canonical set.

Our proof of this result resembles in broad outline Jockusch's proof of Theorem 2.2.22, but requires significant care. We first outline the idea of the proof. For simplicity, assume that $f \colon [\omega]^2 \to \omega$. Using a

$0'$-oracle, we enumerate the complement of a set A, which will be part of a precanonical pair for f. Instead of using an oracle to decide what color to assign to a new element, we blindly assign a color to a new element, hoping that the corresponding thinned set will be infinite, and continue. If we ever discover that the corresponding set is finite using a $0'$-oracle, we change the color, and discard all of the work performed after assigning the bad color.

As long as we proceed through the possible colors intelligently, this outline will work, and will produce an infinite Π^0_2 set A which is part of a precanonical pair. However, if we proceed through the colors naively, we may not be able to extract a Π^0_2 canonical set from A. For example, suppose that we first proceed through the finitely many possible infinitary colors (there are only finitely many because all infinitary colors distinct from the ones assigned to previous elements are equivalent), and then proceed through the finitary colors in increasing order. If at the end of this construction every color is assigned to only finitely many elements of A, it seems impossible to drop elements in the construction to thin out A to a Π^0_2 canonical set. We want to drop elements that repeat earlier colors, but there does not seem to be a way to safely do this since the color at any given stage may change.

We thus carry out the construction in a slightly less intuitively natural manner which will allow us to extract a Π^0_2 canonical set. The idea is to first assign a new element a new infinitary color, then infinitary colors already in use by previous elements, then new finitary colors, and finally finitary colors already in use by previous elements. Of course, there are infinitely many new finitary colors at any stage, so we need a way to determine when to stop and move into used finitary colors. This can be done because the only reason why we reject all of the infinitary colors for a number a is because the set $\{f(a, b) : b \in Z\}$ (where Z is the currently thinned out out we are working inside) is bounded (see Lemma 3.3.3 below), and we can find a bound using a $0'$-oracle. Following this strategy, we will be able to extract a Π^0_2 canonical set from A. For example, if there are infinitely many distinct infinitary colors, we can perform the construction with the additional action of dropping any element from our final set if it ever needs to change color. This will result in a Π^0_2 $\{0, 1\}$-canonical set. On the other hand, if there are finitely many distinct infinitary colors, and one infinitary color d which occurs infinitely often, then for the least such d we can perform the construction, dropping any element from our final set if it ever needs to take on a finitary color or a used infinitary color greater than d. Modulo finitely many mistakes, this will result in a Π^0_2 $\{1\}$-canonical set. The remaining cases are handled in a similar manner.

We now carry out the above sketch in the more general setting of a computable $f\colon [B]^2 \to \omega \times p$ so that we can lift the result to higher exponents.

Theorem 3.3.1. *Suppose that $p \geq 1$, $B \subseteq \omega$ is infinite and computable, and $f\colon [B]^2 \to \omega \times p$ is computable.*

There exists a Π^0_2 set $C \subseteq B$ canonical for f.

Proof. We first use a $0'$-oracle construction to enumerate the complement of an infinite set $A = \{a_0 < a_1 < a_2 < \dots\} \subseteq B$. The construction is a movable marker construction using a $0'$-oracle. We denote by a_i^s the position of the $(i+1)^{\text{st}}$ marker Λ_i at the beginning of stage s. At the beginning of each stage s, we will have a number n^s such that the markers currently having a position are exactly the Λ_i for $i < n^s$, and for each $i < n^s$, we will have numbers e_i^s and q_i^s with $q_i^s < 2p$. Let β^s be the greatest position of any marker up to stage s ($\beta^s = 0$ if $s = 0$), and let $m^s = \max(\{0\} \cup \{\pi_1(f(b_1, b_2)) : b_1 < b_2 \leq \beta^s\})$. Given these and $k \leq n^s$, we say that a number b is *k-acceptable* at s if

- $b \in B$.
- $b > \beta^s$.
- For all $i < k$ with $q_i^s < p$, $f(a_i^s, b) = (e_i^s, q_i^s)$.
- For all $i < k$ with $q_i^s \geq p$, $\pi_2(f(a_i^s, b)) = q_i^s - p$.
- For all $i < k$ with $q_i^s \geq p$, $\pi_1(f(a_i^s, b)) > m^s$.
- For all $i, j < k$ with $q_i^s = q_j^s \geq p$, $f(a_i^s, b) = f(a_j^s, b) \leftrightarrow e_i^s = e_j^s$.

Construction: First set $n^0 = 0$. Stage $s \geq 0$: Assume inductively that we have n^s such that the markers currently having a position are exactly the Λ_i for $i < n^s$, along with e_i^s and q_i^s for all $i < n^s$. Enumerate into $\overline{A}$ all numbers $b \leq \beta^s$ such that $b \neq a_i^s$ for all $i < n^s$. Using a $0'$-oracle, let k^s be the largest $k \leq n^s$ such that there exists a number which is k-acceptable at s. Note that k^s exists because every sufficiently large element of B is 0-acceptable at s. For each $q < 2p$, let $E_q^s = \{e_i^s : i < k^s \text{ and } q_i^s = q\}$.

Case 1: *$k^s = n^s$:* Set $n^{s+1} = n^s + 1$ and place marker Λ_{n^s} on the least k^s-acceptable number. Leave all markers Λ_i with $i < n^s$ in place, and let $e_i^{s+1} = e_i^s$ and $q_i^{s+1} = q_i^s$ for all $i < n^s$. Also, let $q_{n^s}^{s+1} = 2p - 1$ and let $e_{n^s}^{s+1} = \min(\omega - E_{2p-1}^s)$. (Place a new marker, and give it the first new infinitary color in the last column.)

Case 2: *$k^s < n^s$:* Set $n^{s+1} = k^s + 1$ and detach all markers Λ_i with $k^s < i < n^s$. Leave all markers Λ_i with $i \leq k^s$ in place and let $e_i^{s+1} = e_i^s$ and $q_i^{s+1} = q_i^s$ for all $i < k^s$. Let $a^* = a_{k^s}^s$, $e^* = e_{k^s}^s$ and $q^* = q_{k^s}^s$. We now have nine subcases to decide the values $e_{k^s}^{s+1}$ and $q_{k^s}^{s+1}$: (Change a color, column, or both.)

Subcase 2.1: *$q^* \geq p$, $E_{q^*}^s \neq \emptyset$ and $e^* \notin E_{q^*}^s$:* Let $q_{k^s}^{s+1} = q^*$ and $e_{k^s}^{s+1} = \min E_{q^*}^s$. (Take the first used infinitary color for this column.)

Subcase 2.2: *$q^* \geq p$, $e^* \in E_{q^*}^s$, and $e^* \neq \max E_{q^*}^s$:* Let $q_{k^s}^{s+1} = q^*$ and $e_{k^s}^{s+1} = \min\{d \in E_{q^*}^s : d > e^*\}$. (Take the next used infinitary color for this column.)

Subcase 2.3: *$q^* \geq p$ and either $E^s_{q^*} = \emptyset$ or $e^* = \max E^s_{q^*}$:* Let $q^{s+1}_{k^s} = q^* - 1$ and $e^{s+1}_{k^s} = \min(\omega - E^s_{q^*-1})$. (Move either to the next infinitary column, or move to the last finitary column, and assign the first unused color.)

Subcase 2.4: *$q^* < p$, $e^* \notin E^s_{q^*}$, and there exists b which is k^s-acceptable at s with $f(a^*, b) > e^*$:* Let $q^{s+1}_{k^s} = q^*$ and $e^{s+1}_{k^s} = \min\{c \in \omega : c \notin E^s_{q^*}$ and $c > e^*\}$. (Take the next unused finitary color for this column.)

Subcase 2.5: *$q^* < p$, $e^* \notin E^s_{q^*}$, $E^s_{q^*} \neq \emptyset$, and every b which is k^s-acceptable at s satisfies $f(a^*, b) \leq e^*$:* Let $q^{s+1}_{k^s} = q^*$ and $e^{s+1}_{k^s} = \min E^s_{q^*}$. (Take the first used finitary color for this column.)

Subcase 2.6: *$q^* < p$, $e^* \in E^s_{q^*}$, and $e^* \neq \max E^s_{q^*}$:* Let $q^{s+1}_{k^s} = q^*$ and $e^{s+1}_{k^s} = \min\{c \in E^s_{q^*} : c > e^*\}$. (Move to the next used finitary color for this column.)

Subcase 2.7: *$0 < q^* < p$, $E^s_{q^*} = \emptyset$, and every b which is k^s-acceptable at s satisfies $f(a^s_{k^s}, b) \leq e^*$:* Let $q^{s+1}_{k^s} = q^* - 1$ and $e^{s+1}_{k^s} = \min(\omega - E^s_{q^*-1})$. (Move to the next finitary column, and assign the first unused color.)

Subcase 2.8: *$0 < q^* < p$, $e^* \in E^s_{q^*}$, and $e^* = \max E^s_{q^*}$:* Let $q^{s+1}_{k^s} = q^* - 1$ and $e^{s+1}_{k^s} = \min(\omega - E^s_{q^*-1})$. (Move to the next finitary column, and assign the first unused color.)

Subcase 2.9: *Otherwise:* Let $q^{s+1}_{k^s} = 0$ and $e^{s+1}_{k^s} = e^* + 1$. (This case won't occur for any true element of A.)

End Construction.

Claim 3.3.2. *For all $k \in \omega$, each limit $\lim_s a^s_k$, $\lim_s q^s_k$, and $\lim_s e^s_k$ exists, so we may define $a_k = \lim_s a^s_k$, $q_k = \lim_s q^s_k$, and $e_k = \lim_s e^s_k$.*

Proof. We proceed by induction. We assume that the claim is true for all $i < k$ and prove it for k. Let t be the least stage such that for all $i < k$ and all $s \geq t$, we have $a^s_i = a_i$, $q^s_i = q_i$, and $e^s_i = e_i$. At stage t, marker Λ_k is placed on a number b via Case 1 of the construction (since otherwise there exists $i < k$ such that either $q^{t+1}_i \neq q^t_i$ or $e^{t+1}_i \neq e^t_i$), so $n^{t+1} = k + 1$. Since each of a^s_i, q^s_i, and e^s_i for $i < k$ have come to their limits, we must have $k^s \geq k$ and hence $n^s \geq k + 1$ for all $s > t$ by construction (because if $s > t$ is least such that $k^s < k$, then we enter Case 2, so one of $q^s_{k^s}$ or $e^s_{k^s}$ changes). Therefore, by construction, we never again move marker Λ_k, so $a^s_k = a^{t+1}_k$ for all $s \geq t + 1$ and we may let $a_k = \lim_s a^s_k$.

We now show that $\lim_s q^s_k$ and $\lim_s e^s_k$ both exist by showing that $k^s = k$ for only finitely many $s > t$. This suffices, because q^s_k and e^s_k change only at such s. Suppose then that $k^s = k$ for infinitely many $s > t$. Let $Z = \{(d, q) : p \leq q < 2p$ and $d \in E^t_q \cup \{\min(\omega - E^t_q)\}$ Following the construction through the first $|Z|$ many stages $s > t$ with $k^s = k$, we see that for all $(d, q) \in Z$, there is a unique $s_{(d,q)} > t$ such that

$e_k^{s_{(d,q)}} = d$, $q_k^{s_{(d,q)}} = q$, and $k^{s_{(d,q)}} = k$. For each $(d,q) \in Z$, since $k^{s_{(d,q)}} = k$, there are no numbers which are $(k+1)$-acceptable at $s_{(d,q)}$. Let $r_1 = \max\{s_{(d,q)} : (d,q) \in Z\}$. We need the following lemma.

Lemma 3.3.3. *For all $s \geq r_1$, if b is k-acceptable at s, then $\pi_1(f(a_k,b)) \leq m^{r_1}$.*

Proof. Suppose that the lemma is false. Then there exists $s \geq r_1$ and a b which is k-acceptable at s such that $\pi_1(f(a_k,b)) > m^{r_1}$. Let $q = p + \pi_2(f(a_k,b))$. For each d with $(d,q) \in Z$, notice that b is k-acceptable at $s_{(d,q)}$ (since $t \leq s_{(d,q)} \leq r_1 \leq s$), but not $(k+1)$-acceptable at $s_{(d,q)}$. Therefore, for each d with $(d,q) \in Z$, either $\pi_1(f(a_k,b)) \leq m^{s_{(d,q)}} \leq m^{r_1}$, or there exists $i < k$ with $q_i = q$ such that $f(a_i,b) = f(a_k,b) \leftrightarrow e_i \neq d$. Since $\pi_1(f(a_k,b)) > m^{r_1}$, it follows that for all d with $(d,q) \in Z$, there exists $i < k$ with $q_i = q$ such that $f(a_i,b) = f(a_k,b) \leftrightarrow e_i \neq d$. Letting $d = \min(\omega - E_q^t)$, we have $e_i \neq d$ for all $i < k$ with $q_i = q$, so we may choose $j < k$ with $q_j = q$ and $f(a_j,b) = f(a_k,b)$. Letting $d = e_j$, there exists $i < k$ with $q_i = q$ such that $f(a_i,b) = f(a_k,b) \leftrightarrow e_i \neq e_j$. Since $f(a_j,b) = f(a_k,b)$, this implies that $f(a_i,b) = f(a_j,b) \leftrightarrow e_i \neq e_j$, contrary to the fact that b is k-acceptable at s. This is a contradiction, so the proof of the lemma is complete. ☐

We now return to the proof of the claim. Notice that at stage r_1, we set $q_k^{r_1+1} = q = p-1$, so $q_k^{s+1} \leq q_k^s < p$ for all $s > r_1$ by construction. Now, as we continue to follow the construction through stages s with $k^s = k$, we must eventually reach a stage $s > r_1$ with $k^s = k$ such that we do not enter Subcase 2.4 (otherwise, we enter Subcase 2.4 infinitely often, so after m^{r_1} such iterations, we reach an $s \geq r_1$ with $k^s = k$ and $e_k^s \geq m^{r_1}$ where every b which is k-acceptable at s satisfies $\pi_1(f(a_k,b)) \leq m^{r_1} \leq e_k^s$ by Lemma 3.3.3). Let r_2 be the least such stage. If $E_q^t = \emptyset$, then we either enter Subcase 2.7 and set $q_k^{r_2+1} = q-1$ (if $q > 0$), or we enter Subcase 2.9 (if $q = 0$). If $E_q^t \neq \emptyset$, then at stage r_2 we enter Subcase 2.5 and then repeatedly enter Subcase 2.6 whenever $k^s = k$ until we run through all elements of E_q^t, at which point we either enter Subcase 2.8 or Subcase 2.9. Therefore, in either case, we reach a stage $r_3 \geq r_2$ where we either set $q_k^{r_3+1} = q-1$ or we enter Subcase 2.9. Now, the above argument works for the new value of q, so running through each q with $q < p$ in reverse order, we see that we eventually reach a stage r_4 where we enter Subcase 2.9.

Let b be the least number which is k-acceptable at r_4 (such a number exists because otherwise we have $k^{r_4} < k$, which we know is not true). By construction, there exists a stage $s_0 \leq r_4$ such that $e_k^{s_0} = \pi_1(f(a_k,b))$, $q_k^{s_0} = \pi_2(f(a_k,b))$, and $k^{s_0} = k$. Then, b is $(k+1)$-acceptable at s_0, so $k^{s_0} \geq k+1$, a contradiction. It follows that there could not have been infinitely many $s > t$ with $k^s = k$, so the proof of the claim is complete. ☐

Claim 3.3.4. *Let $q < 2p$ be greatest such that $\{k : q_k = q\}$ is infinite.*

(1) Suppose that $q \geq p$ and $\{e_k : q_k = q\}$ is infinite. Then $\{a_k : q_k = q$ and $e_k \neq e_i$ for all $i < k$ with $q_i = q\}$ is a Π^0_2 $\{0,1\}$-canonical set for f.

(2) Suppose that (1) does not hold and $q \geq p$. Then there exists d such that $\{k : q_k = q$ and $e_k = d\}$ is infinite, and for the least such d, the set $\{a_k : q_k = q$ and $e_k = d\}$ is a Π^0_2 $\{1\}$-canonical set for f.

(3) Suppose that $q < p$ and $\{e_k : q_k = q\}$ is infinite. Then $\{a_k : q_k = q$ and $e_k \neq e_i$ for all $i < k$ with $q_i = q\}$ is a Π^0_2 $\{0\}$-canonical set for f.

(4) Suppose that (3) does not hold, but $q < p$. Then there exists c such that $\{k : q_k = q$ and $e_k = c\}$ is infinite, and for the least such c, the set $\{a_k : q_k = q$ and $e_k = c\}$ is a Π^0_2 $\emptyset$-canonical set for f.

Proof. (1) Suppose that $q \geq p$ and $\{e_k : q_k = q\}$ is infinite. Let $C = \{a_k : q_k = q$ and $e_k \neq e_i$ for all $i < k$ with $q_i = q\}$. Notice that C is infinite because $\{e_k : q_k = q\}$ is infinite. To see that C is Π^0_2, perform the above construction, with the additional action of enumerating the number $a^s_{k^s}$ at stage s if either

- $q^s_{k^s} < q$.
- $q^s_{k^s} = q$ and we enter Case 2.

Then a_k is not enumerated if and only if either

- $q_k > q$.
- $q_k = q$ and $e_k \neq e_i$ for all $i < k$ with $q_i = q$.

because at the first s (if any) with $a^s_k = a_k$ and $q^s_k = q$, we set e^s_k to a number different from e_i for all $i < k$ with $q_i = q$, and entrance into Case 2 at any point will result either in $q_k < q$ or $e_k = e_i$ for some $i < k$ with $q_i = q$. Since $\{a_k : q_k > q\}$ is finite, C is Π^0_2 (because removing finitely many elements from a Π^0_2 set leaves a Π^0_2 set). Suppose that $i < k$, $j < l$, $k \leq l$, and $a_i, a_j, a_k, a_l \in C$. Let s be least such that $a^s_l = a_l$. If $k < l$, then a_l is $(\max\{j,k\}+1)$-acceptable at s by construction, hence $f(a_j, a_l) > m^s \geq f(a_i, a_k)$. If $k = l$ and $i \neq j$, then a_k is $(\max\{i,j\}+1)$-acceptable at s, hence $f(a_i, a_k) = f(a_j, a_k) \leftrightarrow e_i = e_j$, so $f(a_i, a_k) \neq f(a_j, a_k)$ because $e_i \neq e_j$. Therefore, $f(a_i, a_k) = f(a_j, a_l) \leftrightarrow i = j$ and $k = l \leftrightarrow a_i = a_j$ and $a_k = a_l$. It follows that C is a Π^0_2 $\{0,1\}$-canonical set for f.

(2) Suppose that (1) does not hold, i.e. $\{e_k : q_k = q\}$ is finite, and $q \geq p$. Let d be least such that $\{k : q_k = q$ and $e_k = d\}$ is infinite, and let $C = \{a_k : q_k = q$ and $e_k = d\}$. To see that C is Π^0_2, perform the above construction, with the additional action of enumerating the number $a^s_{k^s}$ at stage s if either

- $q^s_{k^s} < q$.

- $q^s_{k^s} = q$ and we enter Subcase 2.2 and set $e^{s+1}_{k^s}$ to a number greater than d.

Then a_k is not enumerated if and only if either

- $q_k > q$.
- $q_k = q$ and $e_k \neq e_i$ for all $i < k$ with $q_i = q$.
- $q_k = q$ and $e_k \leq d$.

because at the first s (if any) with $a^s_{k^s} = a_k$ and $q^s_k = q$, we set e^s_k to a new number, after which the value of e^s_k runs through the set $\{e_i : i < k \text{ and } q_i = q\}$ in increasing order until, if ever, we set $q^t_k < q$. Since $\{a_k : q_k > q\} \cup \{a_k : q_k = q, e_k \neq d, \text{ and } e_k \neq e_i \text{ for all } i < k \text{ with } q_i = q\} \cup \{a_k : q_k = q \text{ and } e_k < d\}$ is finite, it follows that C is Π^0_2 (because removing finitely many elements from a Π^0_2 set leaves a Π^0_2 set). Suppose that $i < k$, $j < l$, $k \leq l$, and $a_i, a_j, a_k, a_l \in C$. Let s be least such that $a^s_l = a_l$. If $k < l$, then a_l is $(\max\{j, k\} + 1)$-acceptable at s by construction, hence $f(a_j, a_l) > m^s \geq f(a_i, a_k)$. If $k = l$ then a_k is $(\max\{i, j\} + 1)$-acceptable at s, hence $f(a_i, a_k) = f(a_j, a_k) \leftrightarrow e_i = e_j$, so $f(a_i, a_k) = f(a_j, a_k)$ because $e_i = d = e_j$. Therefore, $f(a_i, a_k) = f(a_j, a_l) \leftrightarrow k = l \leftrightarrow a_k = a_l$. It follows that C is a Π^0_2 $\{1\}$-canonical set for f.

(3) Suppose that $q < p$ and $\{e_k : q_k = q\}$ is infinite. Let $C = \{a_k : q_k = q \text{ and } e_k \neq e_i \text{ for all } i < k \text{ with } q_i = q\}$. Notice that C is infinite because $\{e_k : q_k = q\}$ is infinite. To see that C is Π^0_2, perform the above construction, with the additional action of enumerating the number $a^s_{k^s}$ at stage s if either

- $q^s_{k^s} < q$.
- $q^s_{k^s} = q$ and we enter Subcase 2.5.

Then a_k is not enumerated if and only if either

- $q_k > q$.
- $q_k = q$ and $e_k \neq e_i$ for all $i < k$ with $q_i = q$.

because at the first s (if any) with $a^s_{k^s} = a_k$ and $q^s_k = q$, we initially set e^s_k to a number different from e_i for all $i < k$ with $q_i = q$, and e^t_k will continue to have this property until we either enter into Subcase 2.5 or we set $q^{t+1}_k < q$, at which point e^t_k will never again have this property. Since $\{a_k : q_k > q\}$ is finite, it follows (by removing this finite set) that C is Π^0_2. Suppose that $i < j$ and $a_i, a_j \in C$. Let s be least such that $a^s_j = a_j$. By construction, a_j is $(i + 1)$-acceptable at s, hence $f(a_i, a_j) = (e_i, q_i) = (e_i, q)$. Therefore, whenever $i < k$, $j < l$, and $a_i, a_j, a_k, a_l \in C$, we have $f(a_i, a_k) = f(a_j, a_l) \leftrightarrow (e_i, q) = (e_j, q) \leftrightarrow e_i = e_j \leftrightarrow i = j \leftrightarrow a_i = a_j$. It follows that C is a Π^0_2 $\{0\}$-canonical set for f.

(4) Suppose that (3) does not hold, i.e. $\{e_k : q_k = q\}$ is finite, and $q < p$. Let c be least such that $\{k : q_k = q \text{ and } e_k = c\}$ is infinite, and let $C = \{a_k : q_k = q \text{ and } e_k = c\}$. To see that C is Π^0_2, perform the above construction, with the additional action of enumerating the number $a^s_{k^s}$ at stage s if either

- $q^s_{k^s} < q$.
- $q^s_{k^s} = q$ and we enter Subcase 2.6 and set $e^{s+1}_{k^s}$ to a number greater than c.

Then a_k is not enumerated if and only if either

- $q_k > q$.
- $q_k = q$ and $e_k \neq e_i$ for all $i < k$ with $q_i = q$.
- $q_k = q$ and $e_k \leq c$.

because at the first s (if any) with $a^s_{k^s} = a_k$ and $q^s_k = q$, we initially set e^s_k to a number different from e_i for all $i < k$ with $q_i = q$, and e^t_k will continue to have this property until e^t_k begins to run through $\{e_i : i < k \text{ and } q_i = q\}$ in increasing order until, if ever, we set $q^t_k < q$. Since $\{a_k : q_k > q\} \cup \{a_k : q_k = q, e_k \neq c$, and $e_k \neq e_i$ for all $i < k$ with $q_i = q\} \cup \{a_k : q_k = q \text{ and } e_k < c\}$ is finite, it follows (by removing this finite set) that C is Π^0_2. Suppose that $i < j$ and $a_i, a_j \in C$. Let s be least such that $a^s_j = a_j$. By construction, a_j is $(i+1)$-acceptable at s, hence $f(a_i, a_j) = (e_i, q_i) = (c, q)$. Therefore, whenever $i < k$, $j < l$, and $a_i, a_j, a_k, a_l \in C$, we have $f(a_i, a_k) = (c, q) = f(a_j, a_l)$. It follows that C is a Π^0_2 $\emptyset$-canonical set for f. □

□

Again, using a relativized version of the result for exponent 2 and induction, we can get bounds for higher exponents.

Theorem 3.3.5. *Suppose that $X \subseteq \omega$, $n \geq 2$, $p \geq 1$, $B \subseteq \omega$ is infinite and X-computable, and $f \colon [B]^n \to \omega \times p$ is X-computable. There exists a $\Pi^{0,X}_{2n-2}$ set C canonical for f.*

Proof. We prove the theorem by induction on n. Theorem 3.3.1 relativized to X gives the result for $n = 2$. Suppose that the theorem holds for $n \geq 2$, and that B and $f \colon [B]^{n+1} \to \omega \times p$ are X-computable. By Proposition 3.2.3 relativized to X, there exists a precanonical pair (A, g) for f with $A \oplus g \leq_T X''$. Applying the inductive hypothesis to $g \colon [A]^n \to \omega \times 2p$, there exists $C \subseteq A$ canonical for g such that C is $\Pi^{0,X''}_{2n-2}$. Notice that C is $\Pi^{0,X}_{2n}$. By Claim 3.1.5, C is canonical for f. □

Remark 3.3.6. By Claim 2.3.3, if $n \geq 1$ and $f \colon [\omega]^n \to 2$, then any set C canonical for f is homogeneous for f. Therefore, for each $n \geq 2$, there exists a computable $f \colon [\omega]^n \to 2$ with no Σ^0_n set canonical for f by Theorem 2.2.22. It follows that Theorem 3.3.5 gives a sharp bound in the arithmetical hierarchy for $n = 2$.

Chapter 4

The Regressive Function Theorem and Computability Theory

4.1 h-Regressive Functions

In Chapter 2, we introduced regressive functions and the Regressive Function Theorem. For our purposes, it will be convenient to relax the definition of a regressive function.

Definition 4.1.1. Suppose that $n \geq 1$, $h\colon \omega \to \omega$ and $B \subseteq \omega$ is infinite. A function $f\colon [B]^n \to \omega$ is *h-regressive* if for all $x \in [\omega]^n$, we have $f(x) < h(\min(x))$ whenever $h(\min(x)) > 0$, and $f(x) = 0$ whenever $h(\min(x)) = 0$.

Remark 4.1.2. Notice that a function $f\colon [B]^n \to \omega$ is regressive if and only if it is ι-regressive, where $\iota\colon \omega \to \omega$ is the identity function.

Definition 4.1.3. Suppose that $n \geq 1$, $h\colon \omega \to \omega$, $B \subseteq \omega$ is infinite, and $f\colon [B]^n \to \omega$ is h-regressive. A set M is *minhomogeneous* for f if $M \subseteq B$, M is infinite, and for all $x, y \in [M]^n$ with $\min(x) = \min(y)$ we have $f(x) = f(y)$.

By making very minor changes to the proof of Claim 2.4.4, we obtain the following.

Claim 4.1.4. *Suppose that $n \geq 1$, $h\colon \omega \to \omega$, $B \subseteq \omega$ is infinite, and $f\colon [B]^n \to \omega$ is h-regressive. If $C \subseteq B$ is canonical for f, then C is minhomogeneous for f.*

Therefore, by the Canonical Ramsey Theorem, every h-regressive function has a minhomogeneous set. Although h-regressive functions will be a convenient tool for us, their minhomogeneous sets provide no more complexity than those for regressive functions.

Proposition 4.1.5. *Suppose that $n \geq 1$, $h\colon \omega \to \omega$ is computable, $B \subseteq \omega$ is infinite and computable, and $f\colon [B]^n \to \omega$ is h-regressive and computable. There exists a computable regressive $g\colon [B]^n \to \omega$ such that any set $M \subseteq B$ minhomogeneous for g computes a minhomogeneous set for f.*

Proof. We may assume that h is strictly increasing (otherwise, replace h by the function $h^*\colon \omega \to \omega$ defined by $h^*(0) = h(0)$ and $h^*(k+1) = \max(\{h'(k)+1, h(k+1)\})$, and notice that h^* is computable and that f is h^*-regressive). Define $p\colon \omega \to \omega$ by letting $p(a)$ be the largest $b < a$ such that $h(b)+1 < a$ if there is exists a b with $h(b)+1 < a$, and letting let $p(a) = 0$ otherwise. Notice that p is computable, increasing, and satisfies $\lim_a p(a) = \infty$.

Define $g\colon [B]^n \to \omega$ by setting

$$g(a_1, \ldots, a_n) = \begin{cases} f(p(a_1), \ldots, p(a_n)) + 1 & \text{if } 0 < p(a_1) < \cdots < p(a_n) \\ 0 & \text{otherwise} \end{cases}$$

If $g(a_1, \ldots, a_n) \neq 0$, then $0 < p(a_1) < \cdots < p(a_n)$, hence

$$\begin{aligned} g(a_1, \ldots, a_n) &= f(p(a_1), \ldots, p(a_n)) + 1 \\ &< h(p(a_1)) + 1 \\ &< a_1, \end{aligned}$$

so g is regressive.

Suppose that $M \subseteq B$ is minhomogeneous for g. Suppose that $a_1, a_1' \in M$ with $a_1 < a_1'$ and $0 < p(a_1) = p(a_1')$. Since $\lim_a p(a) = \infty$, there exists $a_2 < a_3 < \cdots < a_n \in M$ such that $a_1' < a_2$ and $0 < p(a_1) = p(a_1') < p(a_2) < p(a_3) < \cdots < p(a_n)$. Since M is minhomogeneous for g, we have

$$\begin{aligned} 0 &= g(a_1, a_1', a_3, \ldots, a_n) \\ &= g(a_1, a_2, a_3, \ldots, a_n) \\ &= f(p(a_1), p(a_2), p(a_3), \ldots, p(a_n)) + 1 \\ &\neq 0, \end{aligned}$$

a contradiction. Hence, if $a, b \in M$ with $a \neq b$ and $0 < p(a), p(b)$, then $p(a) \neq p(b)$.

Since M is infinite, p is increasing and computable, and $\lim_a p(a) = \infty$, it follows that the set $p(M)$ is infinite and $p(M) \leq_T M$. Suppose that $a_1 < \cdots < a_n, b_1 < \cdots < b_n \in M$ with $0 < p(a_1) < \cdots < p(a_n)$,

$0 < p(b_1) < \cdots < p(b_n)$ and $p(a_1) = p(b_1)$. Since $0 < p(a_1) = p(b_1)$, we know from the above that $a_1 = b_1$. Therefore, since M is minhomogeneous for g, we have

$$\begin{aligned} f(p(a_1), \ldots, p(a_n)) + 1 &= g(a_1, \ldots, a_n) \\ &= g(b_1, \ldots, b_n) \\ &= f(p(b_1), \ldots, p(b_n)) + 1, \end{aligned}$$

so $f(p(a_1), \ldots, p(a_n)) = f(p(b_1), \ldots, p(b_n))$. It follows that $p(M)\backslash\{0\}$ is a minhomogeneous set for f which is M-computable. □

4.2 Upper Bounds

Although the Regressive Function Theorem follows immediately from the Canonical Ramsey Theorem, we can obtain better bounds on the Turing degrees and position in the arithmetical hierarchy of minhomogeneous sets for computable f using a direct proof similar to Proof 3 of Ramsey's Theorem in Chapter 2. We follow the outline by defining preminhomogeneous pairs, proving their utility and existence, and then applying induction.

Definition 4.2.1. Suppose that $n \geq 1$, $B \subseteq \omega$ is infinite, and $f\colon [B]^{n+1} \to \omega$ is regressive. We call a pair (A, g) where $A \subseteq B$ is infinite and $g\colon [A]^n \to \omega$, a *preminhomogeneous pair* for f if for all $x \in [A]^n$ and all $a \in A$ with $x < a$, we have $f(x, a) = g(x)$.

Claim 4.2.2. *Suppose that $n \geq 1$, $B \subseteq \omega$ is infinite, $f\colon [B]^{n+1} \to \omega$ is regressive, and (A, g) is a preminhomogeneous pair for f. Then g is regressive, and any $M \subseteq A$ minhomogeneous for g is minhomogeneous for f.*

Proof. Given any $x \in [A]^n$, fix $a \in A$ with $x < a$ and notice that $g(x) = f(x, a) < \min(x)$ if $\min(x) > 0$ and $g(x) = f(x, a) = 0$ if $\min(x) = 0$, so g is regressive. Suppose that $M \subseteq A$ is minhomogeneous for g. Fix $x_1, x_2 \in [M]^n$ and $a_1, a_2 \in M$ with $x_1 < a_1$, $x_2 < a_2$, and $\min(x_1, a_1) = \min(x_2, a_2)$. We have $\min(x_1) = \min(x_2)$, hence $f(x_1, a_1) = g(x_1) = g(x_2) = f(x_2, a_2)$. Therefore, M is minhomogeneous for f. □

Proposition 4.2.3. *Suppose that $n \geq 1$, $B \subseteq \omega$ is infinite and computable, $f\colon [B]^{n+1} \to \omega$ is regressive and computable, and $\mathbf{a} \gg \mathbf{0}'$. There exists a preminhomogeneous pair (A, g) for f such that deg$(A \oplus g) \leq \mathbf{a}$. In particular, there exists a preminhomogeneous pair (A, g) for f such that $(A \oplus g)' \leq_T 0''$.*

Proof. By Theorem 2.2.10 and Lemma 3.2.2, we may fix an r-cohesive set $V \subseteq B$ such that $\deg(V)' \leq \mathbf{a}$. Suppose that $x \in [B]^n$. We have $f(x,a) \leq \min(x)$ for all $a \in B$, so the sets $Z_c = \{a \in B : x < a$ and $f(x,a) = c\}$ for c with $0 \leq c \leq \min(x)$ are computable, pairwise disjoint, and have union $\{a \in B : x < a\}$. Since V is r-cohesive, for each c with $0 \leq c \leq \min(x)$, either $V \cap Z_c$ is finite or $V \cap \overline{Z_c}$ is finite. Therefore, there exists a unique c_x with $0 \leq c_x \leq \min(x)$ such that $V \cap \overline{Z_{c_x}}$ is finite. Moreover, notice that the function from $[B]^n$ to ω given by $x \mapsto c_x$ is V'-computable (since given $x \in [B]^n$, we can run through $b \in B$ in increasing order asking a V'-oracle if all elements of V greater than b lie in a fixed Z_c for some c with $0 \leq c \leq \min(x)$).

We use a V'-oracle to inductively construct a preminhomogeneous pair (A,g) for f. Let $a_0, a_1, \ldots, a_{n-1}$ be the first n elements of V. Suppose that $m \geq n-1$ and we have defined $a_0, a_1, \ldots, a_m$. Using a V'-oracle, let a_{m+1} be the least $b \in V$ such that $b > a_m$ and $f(x,b) = c_x$ for all $x \in [\{a_i : i \leq m\}]^n$ (notice that a_{m+1} exists because $V \subseteq B$ is infinite and $f(x,b) = c_x$ for all sufficiently large $b \in V$). Let $A = \{a_m : m \in \omega\}$ and define $g\colon [A]^n \to \omega$ by $g(x) = c_x$. Then $\deg(A \oplus g) \leq \deg(V)' \leq \mathbf{a}$ and (A,g) is a preminhomogeneous pair for f.

The last statement follow from the fact that there exists $\mathbf{a} \gg \mathbf{0}'$ with $\mathbf{a}' \leq \mathbf{0}''$ by relativizing the Low Basis Theorem to $\mathbf{0}'$. $\square$

Remark 4.2.4. Proposition 4.2.3 can also be proved using an effective analysis of a proof using trees similar to Proof 2 of Proposition 2.2.5.

Theorem 4.2.5. *Suppose that $X \subseteq \omega$, $n \geq 2$, $B \subseteq \omega$ is infinite and X-computable, $f\colon [B]^n \to \omega$ is X-computable, and $\mathbf{a} \gg \deg(X)^{(n-1)}$. There exists a set $M \subseteq B$ minhomogeneous for f such that $\deg(M) \leq \mathbf{a}$.*

Proof. We prove the theorem by induction on n. First, suppose that $n = 2$, B and $f\colon [B]^2 \to \omega$ are X-computable, and $\mathbf{a} \gg \deg(X)'$. By Proposition 4.2.3 relativized to X, there exists a preminhomogeneous pair (A,g) for f with $\deg(A \oplus g) \leq \mathbf{a}$. Since A is trivially minhomogeneous for g, it follows from Claim 4.2.2 that A is minhomogeneous for f.

Suppose that $n \geq 2$ and the theorem holds for n. Suppose that B and $f\colon [B]^{n+1} \to \omega$ are X-computable, and $\mathbf{a} \gg \deg(X)^{(n)}$. By Proposition 4.2.3 relativized to X, there exists a preminhomogeneous pair (A,g) for f with $(A \oplus g)' \leq_T X''$. Applying the inductive hypothesis to $g\colon [A]^n \to \omega$, there exists $M \subseteq A$ minhomogeneous for g with $\deg(M) \leq \mathbf{a}$ since $\mathbf{a} \gg \deg(X)^{(n)} = (\deg(X)'')^{(n-2)} \geq (\deg(A \oplus g)')^{(n-2)} = \deg(A \oplus g)^{(n-1)}$. By Claim 4.2.2, M is minhomogeneous for f. $\square$

We can also use the above results to give bounds on the location of minhomogeneous sets in the arithmetical hierarchy.

Theorem 4.2.6. *Suppose that $X \subseteq \omega$, $n \geq 2$, $B \subseteq \omega$ is infinite and X-computable, and $f \colon [B]^n \to \omega$ is regressive and X-computable. Then f has a $\Pi^{0,X}_n$ minhomogeneous set.*

Proof. We prove the theorem by induction on n. Theorem 3.3.1 relativized to $X \subseteq \omega$ together with Claim 2.4.4 gives the result for $n = 2$. Suppose that we know the theorem for $n \geq 2$, and that $B \subseteq \omega$ is infinite and X-computable, and $f \colon [B]^{n+1} \to \omega$ is regressive and X-computable. By Proposition 4.2.3 relativized to X, there exists a precanonical pair (A, g) for f with $(A \oplus g)' \leq_T X''$. Applying the inductive hypothesis to $g \colon [A]^n \to \omega$, there exists $M \subseteq A$ minhomogeneous for g such that M is $\Pi^{0,A\oplus g}_n$. Then M is $\Pi^{0,(A\oplus g)'}_{n-1}$, so it follows that M is $\Pi^{0,X''}_{n-1}$, and hence $\Pi^{0,X}_{n+1}$. By Claim 4.2.2, M is minhomogeneous for f. ☐

Remark 4.2.7. Theorem 4.2.6 in the case $n = 2$ can also be proved without appealing to Theorem 3.3.1 by using a more natural generalization of the proof of Theorem 2.2.22 in the case $n = 2$.

4.3 Lower Bounds

We next turn our attention to lower bounds, aiming to show that the bounds given by Theorem 4.2.5 and Theorem 4.2.6 are sharp.

Theorem 4.3.1. *There exists a computable regressive $f \colon [\omega]^2 \to \omega$ such that deg$(M) \gg \mathbf{0}'$ for every set M which is minhomogeneous for f.*

Proof. By Proposition 4.1.5, it suffices to find a computable $f \colon [\omega]^2 \to \omega$ and a computable $h \colon \omega \to \omega$ such that f is h-regressive and $\deg(M) \gg \mathbf{0}'$ for every set M which is minhomogeneous for f.

Let $K = \{e : \varphi_e(e)\downarrow\}$ be the usual computably enumerable halting set, and let $\{K_s\}_{s\in\omega}$ be a fixed computable enumeration of K. Define $f_1 \colon \omega^2 \to 2$ by

$$f_1(m,t) = \begin{cases} \varphi^{K_t}_{e,t}(n) & \text{if } m = \langle e,n\rangle \text{ and } \varphi^{K_t}_{e,t}(n)\downarrow \in \{0,1\} \\ 0 & \text{otherwise} \end{cases}$$

Notice that f_1 is computable. Define a computable $f \colon [\omega]^2 \to \omega$ as follows. Given $a, b \in \omega$ with $a < b$, let $f(a,b) = \langle f_1(0,b), f_1(1,b), \ldots, f_1(a,b)\rangle$. Notice that f is h-regressive, where $h \colon \omega \to \omega$ is the computable function given by $h(k) = \max(\{0\} \cup \{\langle a_0, a_1, \ldots, a_k\rangle + 1 : 0 \leq a_i \leq 1 \text{ for } i \leq k\})$.

Suppose that M is a minhomogeneous set for f. For each $e \in \omega$, define $g_e \colon \omega \to 2$ as follows. Given $n \in \omega$, find the least $a_{e,n}, b_{e,n} \in M$ with $\langle e,n\rangle \leq a_{e,n} < b_{e,n}$, and let $g_e(n) = f_1(\langle e,n\rangle, b_{e,n})$. Notice that g_e is M-computable for each $e \in \omega$.

Let $e, n \in \omega$. Since M is minhomogeneous for f, we know that $f(a_{e,n}, b) = f(a_{e,n}, b')$ for all $b, b' \in M$ with $b, b' > a_{e,n}$, so $f_1(\langle e, n\rangle, b) = f_1(\langle e, n\rangle, b')$ for all $b, b' \in M$ with $b, b' > a_{e,n}$. Hence, if $\varphi_e^K(n) \downarrow \in \{0, 1\}$, then $g_e(n) = f_1(\langle e, n\rangle, b_{e,n}) = \varphi_e^K(n)$ because $f_1(\langle e, n\rangle, t) = \varphi_e^K(n)$ for all sufficiently large $t \in \omega$.

Therefore, for all $e \in \omega$, if φ_e^K is $\{0, 1\}$-valued, then g_e is a total M-computable extension. It follows that M computes a total extension of every partial $\{0, 1\}$-valued $\mathbf{0}'$-computable function, hence $\deg(M) \gg \mathbf{0}'$ by Lemma 2.1.6. □

We can use the previous theorem to obtain lower bounds for exponents $n \geq 2$.

Theorem 4.3.2. *For every $X \subseteq \omega$ and $n \geq 2$, there exists an X-computable regressive $f\colon [\omega]^n \to \omega$ such that deg$(M \oplus X) \gg$ deg$(X)^{(n-1)}$ for every set M minhomogeneous for f.*

Proof. We prove the result by induction on n. The case $n = 2$ follows by relativizing Theorem 4.3.1. Suppose that the theorem holds for $n \geq 2$. Fix a X'-computable regressive $g\colon [\omega]^n \to \omega$ such that $\deg(M \oplus X') \gg (\deg(X)')^{(n-1)} = \deg(X)^{(n)}$ for every set M which is minhomogeneous for g. By the Limit Lemma, there exists a computable $g_1\colon [\omega]^{n+1} \to \omega$ such that $\lim_a g_1(x, a) = g(x)$ for all $x \in [\omega]^n$ and $g_1(y) \leq \min(y)$ for all $y \in [\omega]^{n+1}$. By Proposition 2.2.18 relativized to X and the fact that $n + 1 \geq 3$, there exists an X-computable $f_1\colon [\omega]^{n+1} \to 2$ such that for all infinite sets H homogeneous for f_1, we have $f_1([H]^2) = \{0\}$ and $H \oplus X \geq_T X'$. Define an X-computable $f\colon [\omega]^{n+1} \to \omega$ by

$$f(y) = \begin{cases} 0 & \text{if } f_1(y) = 1 \\ g_1(y) + 1 & \text{if } f_1(y) = 0 \end{cases}$$

Notice that $f(y) \leq g_1(y) + 1 \leq \min(y) + 1 < \min(y) + 2$ for all $y \in [\omega]^{n+1}$, hence f is h-regressive, where $h\colon \omega \to \omega$ is the computable function given by $h(k) = k+2$. By Proposition 4.1.5 relativized to X, it suffices to show that $\deg(M \oplus X) \gg \deg(X)^{(n)}$ for all sets M minhomogeneous for f.

Suppose that M is minhomogeneous for f. For each $a \in M$, let $c_a = f(a, x)$ for some (any) $x \in [M]^n$ with $a < x$. Let $Z = \{a \in M : c_a = 0\}$. Since $f_1([Z]^{n+1}) = 1$, it follows that Z is finite. For any $a \in M\backslash Z$, we have $c_a \neq 0$, hence $f_1([M\backslash Z]^{n+1}) = 0$ and $M \oplus X \equiv_T (M\backslash Z) \oplus X \geq_T X'$. Furthermore, for any $x \in [M\backslash Z]^n$ and any $b \in M\backslash Z$ with $x < b$, we have $g_1(x, b)+1 = f(x, b) = c_{\min(x)}$, hence $g(x)+1 = c_{\min(x)}$ for all $x \in [M\backslash Z]^n$. It follows that $M\backslash Z$ is minhomogeneous for g, hence $\deg(M \oplus X) \geq \deg(M \oplus X') \gg \deg(X)^{(n)}$. □

As an immediate corollary of Theorem 4.3.2, we get the following corollary giving a lower bound for the position of minhomogeneous sets in the arithmetical hierarchy.

Corollary 4.3.3. *For every $n \geq 2$, there exists a computable regressive $f\colon [\omega]^n \to \omega$ with no Σ^0_n minhomogeneous set.*

Proof. By Theorem 4.3.2 with $X = \emptyset$, there exists a computable regressive $f\colon [\omega]^n \to \omega$ such that $\deg(M) \gg \mathbf{0}^{(n-1)}$ for every set M minhomogeneous for f. Suppose that M is Σ^0_n minhomogeneous set for f. Let $M_1 \subseteq M$ be an infinite Δ^0_n subset of M, and notice that M_1 is minhomogeneous for f. Since M_1 is Δ^0_n, it follows that $\deg(M_1) \leq \mathbf{0}^{(n-1)}$. Thus, it is not the case that $\deg(M_1) \gg \mathbf{0}^{(n-1)}$, a contradiction. Therefore, there is no Σ^0_n set minhomogeneous for f. □

Remark 4.3.4. Corollary 4.3.3 also follows from the corresponding result for Ramsey's Theorem (Theorem 2.2.22). Fix $f\colon [\omega]^n \to 2$ such that no Σ^0_n set is homogeneous for f. Define $f^*\colon [\omega]^n \to \omega$ by letting $f^*(x) = f(x)$ if $\min(x) \geq 2$ and $f^*(x) = 0$ if $\min(x) < 2$, and notice that f^* is regressive. Suppose that M^* is Σ^0_n and minhomogeneous for f^*. Let M be an infinite Δ^0_n subset of M^* with $0, 1 \notin M$, and notice that M is also minhomogeneous for f^*. Define $g\colon M \to \omega$ by letting $g(a) = g(x)$ for some (any) $x \in [M]^n$ with $a = \min(x)$, and notice that $g \leq_T M$. If $M_0 = \{a \in M : g(a) = 0\}$ is infinite, then M_0 is homogeneous for f and M_0 is Δ^0_n (since $M_0 \leq_T M$), a contradiction. Otherwise, $M_1 = \{a \in M : g(a) = 1\}$ is infinite, so M_1 is homogeneous for f and M_1 is Δ^0_n (since $M_1 \leq_T M$), a contradiction. Therefore, there is no Σ^0_n set minhomogeneous for f^*.

Corollary 4.3.5. *For every $n \geq 2$, there exists a computable regressive $f\colon [\omega]^n \to \omega$ such that every Π^0_n minhomogeneous set M satisfies $\deg(M) \geq \mathbf{0}^{(n)}$.*

Proof. By Theorem 4.3.2 with $X = \emptyset$, there exists a computable regressive $f\colon [\omega]^n \to \omega$ such that $\deg(M) \gg \mathbf{0}^{(n-1)}$ for every set M minhomogeneous for f. If M is a Π^0_n minhomogeneous set for f, then $\deg(M) \gg \mathbf{0}^{(n-1)}$ and $\deg(M)$ is c.e. relative to $\mathbf{0}^{(n-1)}$. Therefore, by the Arslanov Completeness Criterion, $\deg(M) \geq \mathbf{0}^{(n)}$. □

Combining Theorem 4.2.5 and Claim 4.3.2, we obtain the following corollary, analogous to Corollary 2.1.7.

Corollary 4.3.6. *For every $n \geq 2$, there is a "universal" computable regressive $f\colon [\omega]^n \to \omega$, i.e. an f such that given any set M_f minhomogeneous for f and any computable regressive $g\colon [\omega]^n \to \omega$, there exists a set M_g minhomogeneous for g such that $M_g \leq_T M_f$.*

Using Claim 2.4.4, we get similar results for canonical sets for computable $f\colon [\omega]^n \to \omega$.

Corollary 4.3.7. *For every $n \geq 2$, there exists a computable $f\colon [\omega]^n \to \omega$ such that $\deg(C) \gg \mathbf{0}^{(n-1)}$ for every set C canonical for f.*

The next corollary was discussed in Remark 3.3.6, but we also obtain it immediately from Corollary 4.3.3.

Corollary 4.3.8. *For every $n \geq 2$, there exists a computable $f\colon [\omega]^n \to \omega$ with no Σ^0_n canonical set.*

Corollary 4.3.9. *For every $n \geq 2$, there exists a computable $f\colon [\omega]^n \to \omega$ such that every Π^0_n canonical set C satisfies deg$(C) \geq \mathbf{0}^{(n)}$.*

Also, combining Theorem 3.2.4 and Corollary 4.3.7 for $n = 2$, we get the following.

Corollary 4.3.10. *There is a "universal" computable $f\colon [\omega]^2 \to \omega$, i.e. an f such that given any set C_f canonical for f and any computable $g\colon [\omega]^2 \to \omega$, there exists a set C_g canonical for g such that $C_g \leq_T C_f$.*

In contrast, we show in the next chapter that there does not exist a "universal" computable $f\colon [\omega]^2 \to 2$ for Ramsey's Theorem.

4.4 Open Questions

In the previous chapter, we gave upper bounds for canonical sets for computable $f\colon [\omega]^n \to \omega$, in terms of both the Turing degrees and the arithmetical hierarchy. In this chapter, we provided lower bounds. These bounds give sharp characterizations when $n = 2$, but the above upper bounds increase by two jumps for each successive value of n while the lower bounds increase by only one for each successive value of n. In light of Theorem 3.2.6, I conjecture that the upper bounds provided in Theorem 3.2.4 and Theorem 3.3.5 are sharp.

Conjecture 4.4.1. *For every $n \geq 3$, there exists a computable $f\colon [\omega]^n \to \omega$ such that deg$(C) \gg \mathbf{0}^{(2n-3)}$ for every set C canonical for f.*

Conjecture 4.4.2. *For every $n \geq 3$, there exists a computable $f\colon [\omega]^n \to \omega$ with no Σ^0_{2n-2} canonical set.*

Chapter 5

Ramsey Degrees

5.1 Definitions and Basic Results

Definition 5.1.1. Suppose that $n, p \geq 1$, $B \subseteq \omega$ is infinite, and $f\colon [B]^{n+1} \to p$. We say that f is *stable* if $\lim_{b \in B} f(x, b)$ exists for every $x \in [B]^n$.

Stable functions, apart from their intrinsic interest, arise when we restrict f to an r-cohesive subset of B. Therefore, by combining knowledge about r-cohesive sets and homogeneous sets for stable f, we can approach the general question of the complexity of homogeneous sets for arbitrary f. See Corollary A.1.4 for a formal version of this equivalence in second-order arithmetic.

Definition 5.1.2.

(1) A Turing degree $\mathbf{a}$ is *Ramsey* if every computable $f\colon [\omega]^2 \to 2$ has an $\mathbf{a}$-computable homogeneous set.

(2) A Turing degree $\mathbf{a}$ is *s-Ramsey* if every computable stable $f\colon [\omega]^2 \to 2$ has an $\mathbf{a}$-computable homogeneous set.

We first show that the study of stable computable $f\colon [\omega]^2 \to 2$ is equivalent to the study of infinite subsets of Δ^0_2 sets and their complements (or equivalently to the study of arbitrary $0'$-computable $f\colon [\omega]^1 \to 2$).

Claim 5.1.3 (see [12, Proposition 2.1] and [1, Lemma 3.5]).

(1) If $Y \in \Delta^0_2$, then there exists a stable computable $f\colon [\omega]^2 \to 2$ such that for all sets H homogeneous for f, either $H \subseteq Y$ or $H \subseteq \overline{Y}$.

(2) If $f\colon [\omega]^2 \to 2$ is computable and stable, then there exists $Y \in \Delta^0_2$ such that for all infinite sets A satisfying either $A \subseteq Y$ or $A \subseteq \overline{Y}$, A computes an infinite homogeneous set for f.

Therefore, a Turing degree $\mathbf{a}$ *is s-Ramsey if and only if for every Δ^0_2 set Y, there is an infinite* $\mathbf{a}$*-computable subset of either Y or $\overline{Y}$.*

Proof. (1) Fix $Y \in \Delta^0_2$. By the Limit Lemma, there exists a computable $f\colon [\omega]^2 \to 2$ such that $Y(a) = \lim_b f(a,b)$ for all $a \in \omega$. Since $\lim_b f(a,b)$ exists for all $a \in \omega$, f is stable. Suppose that H is an infinite homogeneous set for f. If $f(a,b) = 0$ for all $a, b \in H$ with $a < b$, then for all $a \in H$, we must have $\lim_b f(a,b) = 0$, hence $H \subseteq \overline{Y}$. Similarly, if $f(a,b) = 1$ for all $a, b \in H$ with $a < b$, then for all $a \in A$, we have $\lim_b f(a,b) = 1$, hence $H \subseteq \overline{Y}$.

(2) Let $f\colon [\omega]^2 \to 2$ be computable and stable. Since f is stable, the limit $\lim_b f(a,b)$ exists for every $a \in \omega$, and the set Y defined by $Y(a) = \lim_b f(a,b)$ is Δ^0_2 by the Limit Lemma. Fix an infinite set A such that either $A \subseteq Y$ or $A \subseteq \overline{Y}$. If $A \subseteq Y$, define c_k recursively by letting c_k be the least $a \in A$ such that $c_i < a$ and $f(c_i, a) = 1$ for all $i < k$. Then $\{c_k : k \in \omega\}$ is an A-computable homogeneous set for f. Similarly, if $A \subseteq \overline{Y}$, define c_k recursively by letting c_k be the least $a \in A$ such that $c_i < a$ and $f(c_i, a) = 0$ for all $i < k$. Then $\{c_k : k \in \omega\}$ is an A-computable homogeneous set for f. ◻

To construct a Ramsey degree, we must consider every computable $f\colon [\omega]^2 \to 2$. We first show that it suffices to handle the simpler class of primitive recursive $f\colon [\omega]^2 \to 2$.

Proposition 5.1.4. *If $f\colon [\omega]^2 \to 2$ is computable, then there exists a primitive recursive $g\colon [\omega]^2 \to 2$ such that every set homogeneous for g computes a set homogeneous for f.*

Proof. Fix e such that $f = \varphi_e$. Define $p\colon \omega \to \omega$ by letting $p(s)$ be the greatest $m \le s$ such that $(\forall a \le m)(\forall b \le m)[a < b \to \varphi_{e,s}(a,b)\downarrow]$ (if no such m exists, set $p(s) = 0$). Notice that p is primitive recursive, increasing, and satisfies $\lim_s p(s) = \infty$. Now define a primitive recursive $g\colon [\omega]^2 \to 2$ by

$$g(a,b) = \begin{cases} \varphi_{e,b}(p(a), p(b)) & \text{if } p(a) < p(b) \\ 0 & \text{otherwise} \end{cases}$$

Suppose that H is homogeneous for g. Notice that $p(H)$ is infinite and $p(H) \le_T H$ since p is increasing and $\lim_s p(s) = \infty$. Now for all $a, b \in H$ with $a < b$ and $p(a) < p(b)$, we have $g(a,b) = \varphi_e(p(a), p(b)) = f(p(a), p(b))$. Therefore, for all $c, d \in p(H)$ with $c < d$, there exist $a, b \in A$ with $a < b$ such that $f(c,d) = g(a,b)$. Since H is homogeneous for g, it follows that $p(H)$ is homogeneous for f. ◻

By iterating Seetapun's Theorem (Theorem 2.2.17), we can extend it to a result about Ramsey degrees. Hummel and Jockusch (see [7, Theorem 3.17]) use this technique to prove a version of this result which avoids one cone. We first prove a lemma in the spirit of the Kleene-Post-Spector Theorem on exact pairs.

Lemma 5.1.5. *Let $\{\mathbf{c}_i\}_{i \in \omega}$ be a sequence of degrees with $\mathbf{c}_0 \le \mathbf{c}_1 \le \mathbf{c}_2 \le \dots$. Suppose that $\{\mathbf{b}_k\}_{k \in \omega}$ is a sequence of degrees such that $\mathbf{b}_k \not\le \mathbf{c}_i$ for all $k, i \in \omega$. There exists a degree $\mathbf{a}$ such that $\mathbf{c}_i \le \mathbf{a}$ for all $i \in \omega$*

and $\mathbf{b}_k \not\leq \mathbf{a}$ *for all* $k \in \omega$.

Proof. Let $\langle \cdot \rangle \colon \omega^2 \to \omega$ be a fixed effective bijective coding of pairs of natural numbers. Fix B_k such that $\deg(B_k) = \mathbf{b}_k$ for each $k \in \omega$ and fix C_i such that $\deg(C_i) = \mathbf{c}_i$ for each $i \in \omega$. We build a set A such that $\{a \in \omega : \langle i, a \rangle \in A\} =^* C_i$ for all $i \in \omega$ meeting the requirements

- $R_{\langle e,k \rangle} : \varphi_e^A \neq B_k$.

for all $e, k \in \omega$. We inductively build a sequence $\{f_m\}_{m \in \omega}$ of partial functions such that

- $f_0 \subseteq f_1 \subseteq f_2 \subseteq \dots$
- $\{\langle i, a \rangle : i < m\} \subseteq \mathrm{dom}(f_m)$.
- $\{\langle i, a \rangle : i \geq m\} \cap \mathrm{dom}(f_m)$ is finite.
- $\mathrm{range}(f_m) \subseteq 2$.
- $\{a \in \omega : f_m(\langle i, a \rangle) = 1\} =^* C_i$ for all $i < m$.

Let $f_0 = \emptyset$. Suppose that $m = \langle e, k \rangle$ and we have defined f_m. If there exists $\sigma \in \omega^{<\omega}$ such that

$$(\exists a)[(\forall l < |\sigma|)(f_m(l) \downarrow \to \sigma(l) = f_m(l)) \wedge (\varphi_e^\sigma(a) \downarrow \neq B_k(a))],$$

let σ be the least such (under some canonical ordering), and let $f^* = f_m \cup \sigma$. Otherwise, let $f^* = f_m$. Let $f_{m+1}(b) = b$ for all $b \in \mathrm{dom}(f^*)$ and let $f_{m+1}(\langle m, a \rangle) = C_m(a)$ for all a with $\langle m, a \rangle \notin \mathrm{dom}(f^*)$. Notice that the above invariants are maintained.

Let $A = \bigcup_{m \in \omega} f_m$ and let $\mathbf{a} = \deg(A)$. Since $\{a \in \omega : \langle i, a \rangle \in A\} =^* C_i$ for every $i \in \omega$, it follows that $\mathbf{a} \geq \mathbf{c}_i$ for every $i \in \omega$. Fix $e, k \in \omega$ and let $m = \langle e, k \rangle$. Suppose that R_m is not satisfied, i.e. $\varphi_e^A = B_k$. In the definition of f_{m+1}, it must have been the case that σ did not exist. We show that B_k is f_m-computable. Given a, search using an f_m-oracle until we find the least σ, a, and s (in some canonical ordering) such that $\varphi_{e,s}^\sigma(a) \downarrow$ and $\sigma(l) = f_m(l)$ for all $l < |\sigma|$ with $f_m(l) \downarrow$. Notice that we must have $\varphi_e^\sigma(a) \downarrow = B_k(a)$, so this procedure computes B_k using an f_m-oracle, contrary to the fact that $f_m \equiv_T \oplus_{j<m} C_j$ and $B_k \not\leq_T \oplus_{j<m} C_j$. It follows that R_m is satisfied for every m, hence $\mathbf{b}_k \not\leq \mathbf{a}$ for all $k \in \omega$. □

Proposition 5.1.6. *Suppose that* $f \colon [\omega]^2 \to 2$ *is computable and* $\{\mathbf{b_k}\}_{k \in \omega}$ *is a collection of nonzero degrees. There exists a Ramsey degree* $\mathbf{a}$ *such that* $\mathbf{b_k} \not\leq \mathbf{a}$ *for all* $k \in \omega$.

Proof. Let $\{f_i\}_{i \in \omega}$ be a listing of all computable functions $f \colon [\omega]^2 \to 2$. We first inductively define an increasing sequence of degrees $\{\mathbf{c}_i\}_{i \in \omega}$ such that $\mathbf{b}_k \not\leq \mathbf{c}_i$ for all $k, i \in \omega$. Let $\mathbf{c}_0 = \mathbf{0}$. Suppose that $\mathbf{c}_i$ is defined and $\mathbf{b}_k \nleq \mathbf{c}_i$ for all $k \in \omega$. Since f_i is $\mathbf{c}_i$-computable and $\mathbf{b}_k \nleq \mathbf{c}_i$ for all $k \in \omega$, we may use Seetapun's Theorem relative to $\mathbf{c}_i$ to conclude that there is a set H homogeneous for f_i with $\mathbf{b}_k \nleq \deg(H) \oplus \mathbf{c}_i$ for all $k \in \omega$. Fix such an H and let $\mathbf{c}_{i+1} = \deg(H) \oplus \mathbf{c}_i$.

By Lemma 5.1.5, there exists a degree $\mathbf{a}$ such that $\mathbf{c}_i \leq \mathbf{a}$ for all $i \in \omega$ and $\mathbf{b}_k \nleq \mathbf{a}$ for all $k \in \omega$. For each $i \in \omega$, we have $\mathbf{a} \geq \mathbf{c}_{i+1}$, so there exists and $\mathbf{a}$-computable homogeneous set for f_i. Therefore, $\mathbf{a}$ is Ramsey. $\square$

Corollary 5.1.7. *For any degree $\mathbf{b}$, there exists a Ramsey degree $\mathbf{a}$ with $\mathbf{b} \cap \mathbf{a} = \mathbf{0}$. In particular, there is a minimal pair of Ramsey degrees.*

Proof. Let $\mathbf{b}$ be a degree. Let $\{\mathbf{b}_k\}_{k\in\omega}$ be a listing of all nonzero degrees less than or equal to $\mathbf{b}$. By Proposition 5.1.6, there is a Ramsey degree $\mathbf{a}$ such that $\mathbf{b}_k \nleq \mathbf{a}$ for all $k \in \omega$. We then have $\mathbf{b} \cap \mathbf{a} = \mathbf{0}$. $\square$

5.2 Measure and Category

Sacks (see [24]) proved the following basic results about measure and category of degrees.

Theorem 5.2.1. *Suppose that $\mathbf{a}$ is a nonzero degree. Then $\{\mathbf{b} : \mathbf{b} \geq \mathbf{a}\}$ is meager and has measure* 0.

Using Theorem 5.2.1 and the results from Chapters 2-4, it follows that

(1) For each $n \geq 2$, the set $\{\mathbf{a}$: every computable $f \colon [\omega]^n \to \omega$ has an $\mathbf{a}$-computable canonical set$\}$ is meager and has measure 0.

(2) For each $n \geq 2$, the set $\{\mathbf{a}$: every computable regressive $f \colon [\omega]^n \to \omega$ has an $\mathbf{a}$-computable minhomogeneous set$\}$ is meager and has measure 0.

(3) For each $n \geq 3$, the set $\{\mathbf{a}$: every computable $f \colon [\omega]^n \to 2$ has an $\mathbf{a}$-computable homogeneous set$\}$ is meager and has measure 0.

However, in light of Theorem 2.1.9 and Theorem 5.1.6, we cannot hope to use this method to answer similar questions for $\{\mathbf{a} : \mathbf{a} \gg \mathbf{0}\}$ or the set of Ramsey degrees. Nevertheless, Jockusch and Soare [15, Theorem 5.1 and Corollary 5.4] established that $\{\mathbf{a} : \mathbf{a} \gg \mathbf{0}\}$ is meager and has measure 0, and we proceed to do the same for the set of Ramsey degrees.

Definition 5.2.2. A set A is called *hyperimmune* if it is infinite and there is no computable function f such that

(1) $D_{f(i)} \cap D_{f(j)} = \emptyset$ whenever $i \neq j$.

(2) $D_{f(i)} \cap A \neq \emptyset$ for all i.

where D_i is the $(i+1)^{\text{st}}$ finite set in the usual canonical coding of finite sets by natural numbers.

Proposition 5.2.3 (see [29, Theorem V.2.3]). *An infinite set A is hyperimmune if and only if there is no computable function $h\colon \omega \to \omega$ majorizing p_A.*

We first prove a general lemma which will provide a unified method for the proofs of Proposition 5.2.5 and Proposition 5.3.2.

Lemma 5.2.4. *If $h \leq_T K$, then there exists an infinite and coinfinite $D \in \Delta^0_2$ such that there are infinitely many k with $p_D(k) \geq h(k)$ and there are infinitely many k with $p_{\overline{D}}(k) \geq h(k)$.*

Proof. We build D by finite extensions using a K-oracle, i.e. we produce a sequence of finite binary strings $\sigma_0 \subsetneq \sigma_1 \subsetneq \sigma_2 \subsetneq \dots$ using K such that $D = \bigcup_{m\in\omega} \sigma_m$. We begin by letting $\sigma_0 = \epsilon$. Suppose that we have defined σ_m.

If m is even, let k be the number of ones in σ_m. Using a K-oracle, calculate $h(k)$. If $h(k) \leq |\sigma_m|$, let $\sigma_{m+1} = \sigma_m * 1$, and notice that for any infinite $D \supset \sigma_m$, we have $p_D(k) = |\sigma_m| \geq h(k)$. If $h(k) > |\sigma_m|$, let $\sigma_{m+1} = \sigma_m * 0^{h(k)-|\sigma_m|} * 1$ and notice that for any infinite $D \supset \sigma_m$, we have $p_D(k) = h(k)$.

If m is odd, let k be the number of zeros in σ_m. Using a K-oracle, calculate $h(k)$. If $h(k) \leq |\sigma_m|$, let $\sigma_{m+1} = \sigma_m * 0$, and notice that for any coinfinite $D \supset \sigma_m$, we have $p_{\overline{D}}(n) = |\sigma_m| \geq h(k)$. If $h(k) > |\sigma_m|$, let $\sigma_{m+1} = \sigma_m * 1^{h(k)-|\sigma_m|} * 0$ and notice that for any coinfinite $D \supset \sigma_m$, we have $p_{\overline{D}}(k) = h(k)$.

Since we insert a one in σ_{m+1} when m is even and we insert a zero in σ_{m+1} when m is odd, it follows that there are at least m ones and at least m zeros in σ_{2m+2} for every m. Therefore, D and $\overline{D}$ are infinite. Clearly by construction there are infinitely many k with $p_D(k) \geq h(k)$ and there are infinitely many k with $p_{\overline{D}}(k) \geq h(k)$. □

The next proposition is well-known (for example, let D be a Δ^0_2 1-generic set), but provides a simple application of the above lemma.

Proposition 5.2.5. *There exists $D \in \Delta^0_2$ such that D and $\overline{D}$ are hyperimmune.*

Proof. Fix $h \leq_T K$ such that for every total computable function f, the set $\{k \in \omega : h(k) \geq f(k)\}$ is cofinite (for example, let $h(k) = \max(\{0\} \cup \{\varphi_e(k) : e \leq k \text{ and } \varphi_e(k)\downarrow\})$. By Lemma 5.2.4, there exists an infinite and coinfinite Δ^0_2 set D such that there are infinitely many k with $p_D(k) \geq h(k)$ and there are infinitely many k with $p_{\overline{D}}(k) \geq h(k)$. Thus, if f is a total computable function, it follows that there are infinitely many k with $p_D(k) \geq f(k)+1 > f(k)$ and there are infinitely many k with $p_{\overline{D}}(k) \geq f(k)+1 > f(k)$ (since $k \mapsto f(k)+1$ is computable). Therefore, D and $\overline{D}$ are hyperimmune by Proposition 5.2.3. □

Theorem 5.2.6. *If A is hyperimmune, then $\lambda(\{X \in 2^\omega : X$ computes an infinite subset of $A\}) = 0$.*

Proof. First notice that for each i, the set $\{X \in 2^\omega : \varphi_i^X$ is an infinite subset of $A\}$ is Borel, so the set $\{X \in 2^\omega : X$ computes an infinite subset of $A\} = \bigcup_{i=0}^\infty \{X \in 2^\omega : \varphi_i^X$ is an infinite subset of $A\}$ is also Borel. It follows that all of these sets are measurable.

We prove the contrapositive. Let A be a set such that $\lambda(\{X \in 2^\omega : X$ computes an infinite subset of $A\}) > 0$. We will show that A is not hyperimmune by constructing a computable f satisfying (1) and (2) of Definition 5.2.2. By countable subadditivity of the measure λ, we have that $\lambda(\{X \in 2^\omega : X$ computes an infinite subset of $A\}) \leq \sum_{i=0}^\infty \lambda(\{X \in 2^\omega : \varphi_i^X$ is an infinite subset of $A\})$. Our assumption that $\lambda(\{X \in 2^\omega : X$ computes an infinite subset of $A\}) > 0$ together with the above inequality implies that $\lambda(\{X \in 2^\omega : \varphi_e^X$ is an infinite subset of $A\}) > 0$ for some fixed e. Let $\mathcal{M} = \{X \in 2^\omega : \varphi_e^X$ is an infinite subset of $A\}$ and set $\delta = \lambda\mathcal{M} > 0$. Since $\mathcal{M}$ is a measurable set, we may choose $\sigma_1, \sigma_2, \ldots, \sigma_m \in 2^{<\omega}$ with $\lambda(\mathcal{M} \triangle \bigcup_{i=1}^m \mathcal{I}(\sigma_i)) < \frac{\delta}{3}$. Fix a $\gamma \in \mathbb{Q}$ with $\frac{\delta}{3} < \gamma < \frac{2\delta}{3}$.

We define a computable function f recursively so that $\{D_{f(k)}\}_{k\in\omega}$ witnesses that A is not hyperimmune. Suppose that $k \geq 0$ and we have already defined $f(i)$ for all $i < k$. Let $a_0 = \max(\{0\} \cup \bigcup_{i=0}^{k-1} D_{f(i)})$. Let $Z = \{\tau \in 2^{<\omega} : (\exists i)(\exists a)(\exists s)[1 \leq i \leq m, \tau \supseteq \sigma_i, a > a_0$, and $\varphi_{e,s}^\tau(a) \downarrow= 1)\}$. Search until we find $\tau_1, \tau_2, \ldots, \tau_l \in Z$ with $\lambda(\bigcup_{j=1}^l \mathcal{I}(\tau_j)) > \gamma$. We may carry out this search effectively since we can effectively find an index for Z as a c.e. set and we can effectively find the measure of a finite union of basic open sets.

We now argue that this search must terminate. We first show that $\mathcal{M} \cap \bigcup_{i=1}^m \mathcal{I}(\sigma_i) \subseteq \bigcup_{\tau\in Z} \mathcal{I}(\tau)$. Let $X \in \mathcal{M} \cap \bigcup_{i=1}^m \mathcal{I}(\sigma_i)$. Then φ_e^X is an infinite subset of A and there exists an i with $1 \leq i \leq m$ such that $\sigma_i \subset X$. Thus, there exists an $a > a_0$ with $\varphi_e^X(a) \downarrow= 1$, and since $\sigma_i \subset X$, we may choose an $s \in \omega$ and a $\tau \in \omega^{<\omega}$ such that $\sigma_i \subseteq \tau \subset X$ and $\varphi_{e,s}^\tau(a) \downarrow= 1$. It follows that $\tau \in Z$, and since $\tau \subset X$, we see that $X \in \bigcup_{\tau\in Z} \mathcal{I}(\tau)$.

We have $\lambda(\mathcal{M} \cap \bigcup_{i=1}^m \mathcal{I}(\sigma_i)) + \lambda(\mathcal{M} \backslash \bigcup_{i=1}^m \mathcal{I}(\sigma_i)) = \lambda\mathcal{M} = \delta$, so

$$\begin{aligned}
\lambda(\mathcal{M} \cap \bigcup_{i=1}^m \mathcal{I}(\sigma_i)) &= \delta - \lambda(\mathcal{M} \backslash \bigcup_{i=1}^m \mathcal{I}(\sigma_i)) \\
&\geq \delta - \lambda(\mathcal{M} \triangle \bigcup_{i=1}^m \mathcal{I}(\sigma_i)) \\
&> \delta - \frac{\delta}{3} \\
&= \frac{2\delta}{3} \\
&> \gamma.
\end{aligned}$$

Therefore, $\lambda(\bigcup_{\tau\in Z} \mathcal{I}(\tau)) > \gamma$ from above, so there exists $\tau_1, \tau_2, \ldots, \tau_l \in Z$ with $\lambda(\bigcup_{j=1}^l \mathcal{I}(\tau_j)) > \gamma$. It follows that the search must terminate.

For each j with $1 \leq j \leq l$, effectively find an $a_j > a_0$ with $\varphi_e^{\tau_j}(a_j) \downarrow = 1$. Set $D_{f(k)} = \{a_1, a_2, \ldots, a_l\}$. This ends the definition of $f(k)$, but we use the established notation to show that $D_{f(k)} \cap D_{f(i)} = \emptyset$ for all $i < k$ and that $D_{f(k)} \cap A \neq \emptyset$.

We have $D_{f(k)} \cap D_{f(i)} = \emptyset$ for all $i < k$ since $\max(\{0\} \cup \bigcup_{i=0}^{k-1} D_{f(i)}) = a_0 < D_{f(k)}$. To establish that $D_{f(k)} \cap A \neq \emptyset$, we first show that $\mathcal{M} \cap \bigcup_{j=1}^{l} \mathcal{I}(\tau_j) \neq \emptyset$. Suppose that $\mathcal{M} \cap \bigcup_{j=1}^{l} \mathcal{I}(\tau_j) = \emptyset$. Then $\bigcup_{j=1}^{l} \mathcal{I}(\tau_j) \subseteq (\bigcup_{i=1}^{m} \mathcal{I}(\sigma_i)) \backslash \mathcal{M}$ and so

$$\begin{aligned}
\lambda(\mathcal{M} \triangle \bigcup_{i=1}^{m} \mathcal{I}(\sigma_i)) &\geq \lambda((\bigcup_{i=1}^{m} \mathcal{I}(\sigma_i)) \backslash \mathcal{M}) \\
&\geq \lambda(\bigcup_{j=1}^{l} \mathcal{I}(\tau_j)) \\
&> \gamma \\
&> \frac{\delta}{3}
\end{aligned}$$

a contradiction.

Fix $X \in \mathcal{M} \cap \bigcup_{j=1}^{l} \mathcal{I}(\tau_j)$. Choose j with $1 \leq j \leq l$ such that $\tau_j \subset X$. Since $\varphi_e^{\tau_j}(a_j) \downarrow = 1$, it follows that $\varphi_e^{X}(a_j) \downarrow = 1$, hence $a_j \in A$ because $X \in \mathcal{M}$. Therefore, $a_j \in D_{f(k)} \cap A$, and the proof is complete. □

Remark 5.2.7. The converse of Theorem 5.2.6 is not true. Let A be a noncomputable introreducible set which is not hyperimmune (such a set exists because every nonzero degree is introreducible but there exist nonzero hyperimmune-free degrees). A is not hyperimmune but $\lambda(\{X \in 2^\omega : X$ computes an infinite subset of $A\}) = \lambda(\{\mathbf{b} : \mathbf{b} \geq \deg(A)\}) = 0$. Also, by [15, Theorem 5.5], there exists an immune set A such that $\lambda(\{X \in 2^\omega : X$ computes an infinite subset of $A\}) = 1$. It follows that the collection of sets A such that $\lambda(\{X \in 2^\omega : X$ computes an infinite subset of $A\}) = 0$ strictly contains the collection of hyperimmune sets and is strictly contained in the collection of immune sets.

Corollary 5.2.8. *There exists a computable stable $f\colon [\omega]^2 \to 2$ such that $\lambda(\{X \in 2^\omega : X$ computes a homogeneous set for $f\}) = 0$. In particular, $\lambda(\{\mathbf{a} : \mathbf{a}$ is s-Ramsey$\}) = 0$.*

Proof. By Proposition 5.2.5, there exists $D \in \Delta_2^0$ such that D and $\overline{D}$ are hyperimmune. By Claim 5.1.3, there exists a stable computable $f\colon [\omega]^2 \to 2$ such that for all sets H homogeneous for f, either $H \subseteq D$ or $H \subseteq \overline{D}$. Using Theorem 5.2.6, it follows that $\lambda(\{X \in 2^\omega : X$ computes a homogeneous set for $f\}) = 0$. □

Theorem 5.2.9. *Let A be infinite. A is immune if and only if $\{X \in 2^\omega : X$ computes an infinite subset of $A\}$ is meager.*

Proof. ($\leftarrow$) If A is not immune, then A contains an infinite c.e. subset, and hence an infinite computable subset. Therefore, every set computes an infinite subset of A, so $\{X \in 2^\omega : X$ computes an infinite subset of $A\} = 2^\omega$ is not meager.

($\rightarrow$) Suppose that $\{X \in 2^\omega : X$ computes an infinite subset of $A\}$ is not meager. There exists some e such that the closure of $\{X \in 2^\omega : \varphi_e^X$ is an infinite subset of $A\}$ contains a nonempty open set (otherwise, each such set is nowhere dense, so $\{X \in 2^\omega : X$ computes an infinite subset of $A\}$ would be meager since it is the countable union of these sets). Fix such an e and let $\mathcal{M} = \{X \in 2^\omega : \varphi_e^X$ is an infinite subset of $A\}$. Since $\mathrm{cl}(\mathcal{M})$ (the closure of $\mathcal{M}$) contains an open set, we may choose $\sigma \in 2^{<\omega}$ with $\mathcal{I}(\sigma) \subseteq \mathrm{cl}(\mathcal{M})$. Notice that given any nonempty open set $\mathcal{O} \subseteq \mathrm{cl}(\mathcal{M})$, we must have $\mathcal{O} \cap \mathcal{M} \neq \emptyset$.

Let $C = \{k \in \omega : (\exists \tau)(\exists s)[\tau \supseteq \sigma \wedge \varphi_{e,s}^\tau(k) \downarrow = 1]\}$. Note that C is infinite and c.e. (C is infinite because there exists $X \in \mathcal{I}(\sigma) \cap \mathcal{M}$ by the above comments). Let $k \in C$. Choose $\tau \supseteq \sigma$ with $\varphi_e^\tau(k) \downarrow = 1$. If $k \notin A$, then $\mathcal{I}(\tau)$ would be a nonempty open subset of $\mathcal{I}(\sigma) \subseteq \mathrm{cl}(\mathcal{M})$ such that $\mathcal{I}(\tau) \cap \mathcal{M} = \emptyset$, contrary to the above remarks. Therefore $k \in A$. It follows that that $C \subseteq A$, and hence A is not immune. □

Corollary 5.2.10. *There exists a computable stable $f \colon [\omega]^2 \to 2$ such that $\{X \in 2^\omega : X$ computes a homogeneous set for $f\}$ is meager. In particular, $\{\mathbf{a} : \mathbf{a}$ is s-Ramsey$\}$ is meager.*

Proof. Proof. By Proposition 5.2.5, there exists $D \in \Delta_2^0$ such that D and $\overline{D}$ are hyperimmune. By Claim 5.1.3, there exists a stable computable $f \colon [\omega]^2 \to 2$ such that for all sets H homogeneous for f, either $H \subseteq D$ or $H \subseteq \overline{D}$. Using Theorem 5.2.9, it follows that $\{X \in 2^\omega : X$ computes a homogeneous set for $f\}$ is meager. □

□

5.3 s-Ramsey Degrees

Definition 5.3.1 (Downey, Jockusch, Stob [3]). A degree $\mathbf{a}$ is *array noncomputable* (abbreviated *ANC*) if for each $h \leq_{wtt} K$, there is an $\mathbf{a}$-computable function g such that $g(k) \geq h(k)$ for infinitely many k.

Proposition 5.3.2. *There exists $B \in \Delta_2^0$ such that deg(A) is ANC for every infinite set A with either $A \subseteq B$ or $A \subseteq \overline{B}$.*

Proof. Fix a computable enumeration $\{K_s\}_{s \in \omega}$ of K. Define a function $h \colon \omega \to \omega$ by letting $h(k)$ be the least s such that $K \restriction k = K_s \restriction k$, and notice that $h \leq_T K$. By Lemma 5.2.4, there exists $D \in \Delta_2^0$ such that there are infinitely many k with $p_D(k) \geq h(k)$ and there are infinitely many k with $p_{\overline{D}}(k) \geq h(k)$. Suppose that A is an infinite set such that either $A \subseteq D$ or $A \subseteq \overline{D}$. Since $p_A \leq_T A$ and either $p_A(k) \geq p_D(k)$ for all k or $p_A(k) \geq p_{\overline{D}}(k)$ for all k, it follows from [3, Theorem 1.3] that deg(A) is ANC. $\square$

Corollary 5.3.3 (see [3, Theorem 2.1]). *If* $\mathbf{a}$ *is s-Ramsey, then* $\mathbf{a}$ *bounds a 1-generic degree. In particular,* $\mathbf{a}$ *is not minimal.*

Remark 5.3.4. At the end of [3], Downey, Jockusch, and Stob remark that the ANC degrees have measure 0 (using a result of Kurtz). Hence, we get another proof that the s-Ramsey degrees have measure 0.

The following result gives more information about the complexity of s-Ramsey degrees.

Theorem 5.3.5 (Downey, Hirschfeldt, Lempp, Solomon [2]). *There exists* $D \in \Delta^0_2$ *such that every infinite set* A *with either* $A \subseteq D$ *or* $A \subseteq \overline{D}$ *is not low.*

Corollary 5.3.6. *There does not exist a low s-Ramsey degree.*

We extend Corollary 5.3.6 in two directions, first by showing that the only Δ^0_2 s-Ramsey degree is $\mathbf{0}'$, and then that there is no low$_2$ s-Ramsey degree.

Theorem 5.3.7. *The only* Δ^0_2 *s-Ramsey degree is* $\mathbf{0}'$.

Proof. Suppose that D is a Δ^0_2 set such that $K \not\leq_T D$. We will construct a Δ^0_2 set B such that D does not compute any infinite subset of B or $\overline{B}$. We have the following requirements:

- R_{2e}: If φ^D_e is $\{0,1\}$-valued, total, and infinite, then there is an a with $\varphi^D_e(a) = 1$ and $B(a) = 0$.
- R_{2e+1}: If φ^D_e is $\{0,1\}$-valued, total, and infinite, then there is an a with $\varphi^D_e(a) = 1$ and $B(a) = 1$.

For each e, define a partial function u_e by letting $u_e(a)$ be the use of φ^D_e on input a if $\varphi^D_e(a)\downarrow$ and letting $u_e(a)\uparrow$ otherwise. Also, define a computable partial function θ by letting $\theta(a) = (\mu t)[a \in K_t]$ if $a \in K$ and $\theta(a)\uparrow$ otherwise.

The idea of the proof is as follows. Suppose that $i = 2e + d$ with $0 \leq d \leq 1$. If φ^D_e is $\{0,1\}$-valued, total, and infinite, we need to exhibit an a with $\varphi^D_e(a) = 1$ and $B(a) = d$. We work on requirement R_i by finding an a such that our approximation to the computation $\varphi^D_e(a)$ at stage s produces a 1, claiming the least such available a, and ensuring that $B_s(a) = d$. Since we need to ensure that the computable approximation $\{B_s\}_{s\in\omega}$ settles down on each a, we must temper this strategy in order to prevent two conflicting requirements from claiming the same a at infinitely many stages. Therefore, we impose the additional requirement that in order for R_i to claim a at s, we must have $a \geq i$ and the current approximate use of the computation $\varphi^D_e(a)$ is less than the maximum of the approximations to $\theta(b)$ for all $b \leq a$. Now if $\varphi^D_e(a)\downarrow\neq 1$ or $\varphi^D_e(a)\uparrow$, then R_i will claim a at only finitely many stages s (in the latter case this follows because the value of the approximation of the use must tend to infinity). If $\varphi_e^D(a) = 1$, then the set of stages at which R_i claims a will either be finite or cofinite, depending on the size of the use of the computation. Thus, each requirement will have only a finite set of values of a which it is unable to eventually claim and furthermore the approximation $\{B_s\}_{s\in\omega}$ will eventually settle down on each a. Now suppose that φ_e^D is $\{0,1\}$-valued, total, and infinite. We need to argue that requirement R_i succeeds in claiming an a with $\varphi_e^D(a) = 1$ on a cofinite set of stages. The only way this could fail is if for almost all a with $\varphi_e^D(a) = 1$, the use of the computation exceeds $\max\{\theta(b) : b \leq a \text{ and } \theta(b)\downarrow\}$. However, we can argue that such a situation would imply the existence of a D-computable function h such that for all k, we have $k \in K \leftrightarrow k \in K_{h(k)}$, contrary to the hypothesis that $K \not\leq_T D$.

Fix a computable approximation $\{D_s\}_{s\in\omega}$ of D. Given e and s, let $u_{e,s}(a)$ be the use of $\varphi_{e,s}^{D_s}$ on input a if $\varphi_{e,s}^{D_s}(a)\downarrow$ and let $u_{e,s}(a)\uparrow$ otherwise. Also, given s, let $\theta_s(a) = (\mu t \leq s)[a \in K_t]$ (if $a \notin K_s$, then $\theta_s(a)\uparrow$).

We give a computable approximation $\{B_s\}_{s\in\omega}$ of the set B. At stage s, proceed as follows. We run through substages i in increasing order from $i = 0$ to $i = s$ working on requirement R_i. At substage $i = 2e + d$ with $0 \leq d \leq 1$, let $a_{i,s}$ be the least $a \leq s$, if it exists, such that

(1) $a \geq i$.

(2) $\varphi_{e,s}^{D_s}(a) = 1$.

(3) $u_{e,s}(a) < \max\{\theta_s(b) : b \leq a \text{ and } \theta_s(b)\downarrow\}$.

(4) a has not yet been claimed at stage s.

If $a_{i,s}$ exists, set $B_s(a_{i,s}) = d$, say that requirement R_i claims $a_{i,s}$ at stage s, and move to the next substage. If $a_{i,s}$ does not exist, simply move to the next substage. At the end of the substages, conclude stage s by setting $B_s(a) = 0$ for each a which was not claimed during stage s.

For each $i, a \in \omega$, let $Z_{i,a} = \{s \in \omega : R_i \text{ claims } a \text{ at stage } s\}$. We prove the following for each $i, a \in \omega$ (where $i = 2e + d$ with $0 \leq d \leq 1$):

(1) If $\varphi_e^D(a)\downarrow\neq 1$, then $Z_{i,a}$ is finite.

(2) If $\varphi_e^D(a)\uparrow$, then $Z_{i,a}$ is finite.

(3) If $\varphi_e^D(a)\downarrow= 1$, then $Z_{i,a}$ is either finite or cofinite.

(4) $\lim_s B_s(a)$ exists.

Fix a. We first prove items (1), (2), and (3) by induction on i. Suppose that $i = 2e + d$ with $0 \le d \le 1$ and that for all $j < i$, the set $Z_{j,a}$ is either finite or cofinite. First notice that $Z_{i,a} = \emptyset$ if $a < i$, so we may assume that $a \ge i$. If there exists $j < i$ such that $Z_{j,a}$ is cofinite, then $Z_{i,a}$ is finite because at most one requirement may claim a at a given stage. Suppose that $Z_{j,a}$ is finite for all $j < i$. Fix t_0 such that for all $j < i$ and all $s \ge t_0$, R_j does not claim a at stage s. Let $m = \max(\{0\} \cup \{\theta(b) : b \le a \text{ and } \theta(b)\downarrow\}$ and fix $t_1 \ge t_0$ such that $\theta_s(b)\downarrow= \theta(b)$ for all $b \le a$ with $\theta(b)\downarrow$.

Suppose first that $\varphi_e^D(a)\downarrow\ne 1$. Fix $t_2 \ge t_1$ such that $\varphi_{e,s}^{D_s}(a)\downarrow\ne 1$ for all $s \ge t_2$. Then for all $s \ge t_2$, requirement R_i does not claim a at stage s. Hence, $Z_{i,a}$ is finite.

Suppose that $\varphi_e^D(a)\uparrow$. Fix $t_2 \ge t_1$ such that $D(b) = D_s(b)$ for all $b \le m$ and all $s \ge t_2$. Then for all $s \ge t_2$, if $\varphi_{e,s}^{D_s}(a) = 1$, we must have $u_{e,s}(a) > m = \max\{\theta_s(b) : b \le a \text{ and } \theta_s(b)\downarrow\}$ because otherwise $\varphi_e^D(a) = 1$. It follows that for all $s \ge t_2$, requirement R_i does not claim a at stage s. Hence, $Z_{i,a}$ is finite.

Suppose now that $\varphi_e^D(a) = 1$ and $u_e(a) \ge m$. Fix $t_2 \ge t_1$ such that for all $s \ge t_2$, we have $\varphi_{e,s}^{D_s}(a) = 1$ and $D(b) = D_s(b)$ for all $b \le u_e(a)$. Then for all $s \ge t_2$, requirement R_i does not claim a at stage s because $u_{e,s}(a) \ge m = \max\{\theta_s(b) : b \le a \text{ and } \theta_s(b)\downarrow\}$. Hence, $Z_{i,a}$ is finite.

Finally, suppose that $\varphi_e^D(a) = 1$ and $u_e(a) < m$. Fix $t_2 \ge t_1$ such that for all $s \ge t_2$, we have $\varphi_{e,s}^{D_s}(a) = 1$ and $D(b) = D_s(b)$ for all $b \le u_e(a)$. Then for all $s \ge t_2$, requirement R_i claims a at stage s. Hence, $Z_{i,a}$ is cofinite.

Therefore, for each i, the set $Z_{i,a} = \{s \in \omega : R_i \text{ claims } a \text{ at stage } s\}$ is either finite or cofinite. Item (4) now follows because $Z_{i,a} = \emptyset$ if $i > a$, hence $\lim_s B_s(a) = 0$ if $Z_{i,a}$ is finite for all $i \le a$ and $\lim_s B_s(a) = d$ if $i = 2e + d$ with $0 \le d \le 1$ is such that $Z_{i,a}$ is cofinite. This ends the proof of (1)-(4).

Define B by letting $B(a) = \lim_s B_s(a)$, and notice that B is Δ_2^0 by the Limit Lemma. We end the proof by showing that each requirement R_i is satisfied. Suppose that $i = 2e + d$ with $0 \le d \le 1$, and that φ_e^D is $\{0,1\}$-valued, total, and infinite. Let $A = \{a \in \omega : a \ge i \text{ and } \varphi_e^D(a) = 1\}$, and let $y = \{a \in A : Z_{j,a} \text{ is cofinite for some } j < i\}$. Notice that $A \setminus y \le_T D$ since $A \le_T D$ and y is finite (for each $j < i$, there is at most one a with $Z_{j,a}$ cofinite). Define $h\colon \omega \to \omega$ as follows. Given $k \in \omega$, let a_k be the least element of $A \setminus y$ greater than or equal to k, and let $h(k) = u_e(a_k)$. Since $u_e \le_T D$, it follows that $h \le_T D$.

If for all $k \in \omega$ we have $k \in K \leftrightarrow k \in K_{h(k)}$, then $K \le_T h \le_T D$, contrary to hypothesis. Thus, we may let $l = \min\{k \in \omega : k \in K \setminus K_{h(k)}\}$. We then have

(1) $a_l \ge l$.

(2) $a_l \ge i$.

(3) $\varphi_e^D(a_l) = 1$.

(4) $\theta(l) > h(l) = u_e(a_l)$.

(5) For all $j < i$, there exists t such that R_j does not claim a_l at any stage $s \geq t$.

Therefore, we may fix $t \geq a_l$ such that for all $s \geq t$, we have $\varphi_{e,s}^{D_s}(a_l) = 1$, $\theta_s(l) = \theta(l)$, $u_{e,s}(a_l) = u_e(a_l)$, and for each $j < i$, R_j does not claim a_l at stage s. Thus, for every $s \geq t$, requirement R_i claims some $b \leq a_l$ at stage s. Since $Z_{i,b}$ is either finite or cofinite for each $b \leq a_l$, it follows that $Z_{i,b}$ is cofinite for exactly one $b \leq a_l$. By the above argument, we must have $\varphi_e^D(b) = 1$, and by construction, $B(b) = d$. Therefore, requirement R_i is satisfied. ☐

Theorem 5.3.7 extends Corollary 5.3.6, but it does not provide a fixed stable computable $f\colon [\omega]^2 \to 2$ such that all Δ_2^0 homogeneous sets for f have degree $\mathbf{0}'$. We are thus left with the following question, which is the corresponding extension of Theorem 5.3.5.

Question 5.3.8. *Does there exists a Δ_2^0 set D such that every infinite Δ_2^0 set A with either $A \subseteq D$ or $A \subseteq \overline{D}$ has degree $\mathbf{0}'$?*

5.4 s-Ramsey Degrees and Universality

Recall from Theorem 2.2.11 that given a computable $f\colon [\omega]^2 \to 2$ and an $\mathbf{a} \gg \mathbf{0}'$, there exists a set H homogeneous for f with $\deg(H)' \leq \mathbf{a}$. By the Low Basis Theorem relative to $\mathbf{0}'$ there exists an $\mathbf{a} \gg \mathbf{0}'$ such that $\mathbf{a}' = \mathbf{0}''$. Using this $\mathbf{a}$ in the above yields the following corollary.

Corollary 5.4.1 (Cholak, Jockusch, Slaman [1]). *Suppose that $f\colon [\omega]^2 \to 2$ is computable. There exists a set H homogeneous for f with $H'' \leq_T 0''$.*

We now proceed to show that there is no fixed degree $\mathbf{a}$ with $\mathbf{a}'' \leq \mathbf{0}''$ such that every computable (stable) $f\colon [\omega]^2 \to 2$ has an $\mathbf{a}$-computable homogeneous set, i.e. there is no low$_2$ s-Ramsey degree.

Theorem 5.4.2. *Suppose $f\colon \omega^2 \to 2$ satisfies $f'' \leq_T 0''$. For each e, let $Z_e = \{a \in \omega : f(e,a) = 1\}$. There exists $D \in \Delta_2^0$ such that for all e, if Z_e is infinite, then $Z_e \nsubseteq D$ and $Z_e \nsubseteq \overline{D}$.*

Proof. We give a K-oracle construction of the set D. By the Recursion Theorem relative to K, we may assume that we have an index d such that $D = \varphi_d^K$. We have the following requirements:

- R_{2e} : Z_e is finite or $(\exists a)[f(e,a) = 1$ and $D(a) = 0]$.
- R_{2e+1} : Z_e is finite or $(\exists a)[f(e,a) = 1$ and $D(a) = 1]$.

We can rewrite these requirement as:

- R_{2e} : $(\exists m)(\forall a)[a \geq m \rightarrow f(e,a) = 0] \vee$

  $(\exists a)[f(e,a) = 1 \wedge (\exists t)(\forall s \geq t)[\varphi_{d,s}^{K_s}(a) = 0]]$.

- R_{2e+1} : $(\exists m)(\forall a)[a \geq m \rightarrow f(e,a) = 0] \vee$

  $(\exists a)[f(e,a) = 1 \wedge (\exists t)(\forall s \geq t)[\varphi_{d,s}^{K_s}(a) = 1]]$.

Notice that each of these requirements is Σ_2^f, and furthermore we can effectively find an index for each as such. Therefore, for each i, we can effectively find an m_i such that R_i is satisfied if and only if $m_i \in f''$. By the Limit Lemma relative to K and the fact that $f'' \leq_T 0'' \equiv_T K'$, there exists a function $g \colon \omega^2 \rightarrow 2$ with $g \leq_T K$ such that for all m, we have $m \in f'' \leftrightarrow \lim_s g(m,s) = 1$ and $m \notin f'' \leftrightarrow \lim_s g(m,s) = 0$. Furthermore, an index l such that $g = \varphi_l^K$ may be found effectively. Notice that, for all i, R_i is satisfied if and only if $\lim_s g(m_i, s) = 1$.

We now give our K-oracle construction of the set D in stages, defining $D(s)$ at stage s. Given s, find the least $i \leq s$, if it exists, such that $g(m_i, s) = 0$, and call it i_s. If i_s does not exist, let $D(s) = 0$. Otherwise, if i_s is even, let $D(s) = 0$, and if i_s is odd, let $D(s) = 1$. This ends the construction.

We now verify that R_i is satisfied for all i by induction. Suppose that R_j is satisfied for all $j < i$, but R_i is not satisfied. There exists $t \geq i$ such that $g(m_j, s) = 1$ for all $j < i$ and $g(m_i, s) = 0$ whenever $s \geq t$. Hence, by construction, $i_s = i$ for all $s \geq t$. Therefore, if i is even, say $i = 2e$, then $D(s) = 0$ for all $s \geq t$, so R_i is satisfied (if Z_e is infinite, then there exists $a \geq t$ with $f(e,a) = 1$), a contradiction. Similarly, if i is odd, say $i = 2e+1$, then $D(s) = 1$ for all $s \geq t$, so R_i is satisfied, a contradiction. □

The following definition and proposition are essentially due to Jockusch [11].

Definition 5.4.3. Let $\mathbf{a}$ be a Turing degree and $\mathcal{C}$ a class of functions. We say that the class $\mathcal{C}$ is **a**-*subuniform* if there exists an $\mathbf{a}$-computable $f \colon \omega^2 \rightarrow \omega$ such that for every $g \in \mathcal{C}$, there exists $e \in \omega$ such that $g(a) = f(e,a)$ for all $a \in \omega$.

Proposition 5.4.4. *Suppose that $\mathbf{a} \gg \mathbf{0}$ and $h \colon \omega \rightarrow \omega$ is computable. Let $\mathcal{C} = \{g : g$ is a computable function with $g(a) \leq h(a)$ for all $a \in \omega\}$. Then $\mathcal{C}$ is $\mathbf{a}$-subuniform. Furthermore, an f witnessing that $\mathcal{C}$ is $\mathbf{a}$-subuniform may be chosen such that $f(e,a) \leq h(a)$ for all $e, a \in \omega$.*

Proof. Let T be the set of all $\sigma \in \omega^{<\omega}$ such that

- $\sigma(\langle e,a \rangle) \leq h(a)$ for all e, a with $\langle e,a \rangle < |\sigma|$.

- $\sigma(\langle e,a \rangle) = \varphi_{e,|\sigma|}(a)$ for all e, a with $\langle e,a \rangle < |\sigma|$ such that $\varphi_{e,|\sigma|}(a) \downarrow \leq h(a)$.

- $\sigma(j) = 0$ if $j < |\sigma|$ with $j \neq \langle e,a \rangle$ for all e, a.

Notice that T is a computable infinite tree which is bounded by the computable function $m \mapsto h((m)_1)$. Since $\mathbf{a} \gg \mathbf{0}$, T has an $\mathbf{a}$-computable branch f^*. Define a computable $f\colon \omega^2 \to \omega$ by letting $f(e,a) = f^*(\langle e,a\rangle)$. Suppose that $g \in \mathcal{C}$ and fix $e \in \omega$ with $g = \varphi_e$. Let $a \in \omega$, and suppose $g(a) \neq f(e,a)$ so that $\varphi_e(a) \neq f^*(\langle e,a\rangle)$. If we let $s \in \omega$ be such that $\langle e,a\rangle < s$ and $\varphi_{e,s}(a)\downarrow$, then $f^* \upharpoonright s \notin T$, a contradiction. If follows that $g(a) = f(e,a)$ for all $a \in \omega$. $\square$

Corollary 5.4.5. *If $X'' \leq_T 0''$, then there exists $D \in \Delta^0_2$ such that X does not compute any infinite subset of D or $\overline{D}$.*

Proof. Relativizing Proposition 5.4.4 and the Low Basis Theorem to X, there exists $f\colon \omega^2 \to 2$ low over X such that for every X-computable set Z, there exists an $e \in \omega$ with $Z = \{a \in \omega : f(e,a) = 1\}$. We have $f'' = (f')' \equiv_T (X')' = X'' \leq_T 0''$. Therefore, by Theorem 5.4.2, there exists $D \in \Delta^0_2$ such that for all infinite X-computable sets A, $A \nsubseteq D$ and $A \nsubseteq \overline{D}$. $\square$

Corollary 5.4.6. *There is no s-Ramsey degree $\mathbf{a}$ satisfying $\mathbf{a}'' \leq \mathbf{0}''$.*

Corollary 5.4.7. *Suppose that $f\colon [\omega]^2 \to 2$ is computable. There exists a set H_f homogeneous for f and a computable stable $g\colon [\omega]^2 \to 2$ such that $H_g \nleq_T H_f$ for all sets H_g homogeneous for g.*

Proof. Suppose that $f\colon [\omega]^2 \to 2$ is computable. By Theorem 2.2.11 and the Low Basis Theorem relative to $\mathbf{0}'$, there exists H_f homogeneous for f such that $H_f'' \leq_T 0''$. By Corollary 5.4.6, $\deg(H_f)$ is not s-Ramsey, so there exists a computable stable $g\colon [\omega]^2 \to 2$ such that $H_g \nleq_T H_f$ for all sets H_g homogeneous for g. $\square$

Corollary 5.4.8.

(1) There is no "universal" computable $f\colon [\omega]^2 \to 2$, i.e. a computable $f\colon [\omega]^2 \to 2$ such that for all computable $g\colon [\omega]^2 \to 2$ and all sets H_f homogeneous for f, there is a set H_g homogeneous for g such that $H_g \leq_T H_f$.

(2) There is no "universal" computable stable $f\colon [\omega]^2 \to 2$, i.e. a computable stable $f\colon [\omega]^2 \to 2$ such that for all computable stable $g\colon [\omega]^2 \to 2$ and all sets H_f homogeneous for f, there is a set H_g homogeneous for g such that $H_g \leq_T H_f$.

5.5 Open Questions

A fundamental question about the relation between Ramsey's Theorem for exponent 2 and König's Lemma is the following.

Question 5.5.1. *Does there exist a (stable) computable $f\colon [\omega]^2 \to 2$ such that deg(H) $\gg \mathbf{0}$ for all sets H homogeneous for f? Does every (s-)Ramsey degree $\mathbf{a}$ satisfy $\mathbf{a} \gg \mathbf{0}$?*

Hummel and Jockusch [7, Theorem 2.10] proved that there exists a computable $f\colon [\omega]^2 \to 2$ such that H is effectively immune relative to $0'$ for all sets H homogeneous for f. The following strengthening of this result remains open.

Question 5.5.2. *Does there exist a computable $f\colon [\omega]^2 \to 2$ such that deg(H)$\cup\mathbf{0}' \gg \mathbf{0}'$ for all sets H homogeneous for f? Does every Ramsey degree $\mathbf{a}$ satisfy $\mathbf{a} \cup \mathbf{0}' \gg \mathbf{0}'$?*

Roughly, Corollary 5.4.8 says that we can not combine all computable $f\colon [\omega]^2 \to 2$ into a single universal $g\colon [\omega]^2 \to 2$. The following question about combining two computable 2-colorings into one remains open.

Question 5.5.3. *Do there exist computable $f_1\colon [\omega]^2 \to 2$ and $f_2\colon [\omega]^2 \to 2$ such there is no computable $g\colon [\omega]^2 \to 2$ with the property that every set H_g homogeneous for g computes both a set H_{f_1} homogeneous for f_1 and a set H_{f_2} homogeneous for f_2?*

Remark 5.5.4. The number of colors in the above g is crucial. If $f_1\colon [\omega]^2 \to 2$ and $f_2\colon [\omega]^2 \to 2$ are computable, then $g\colon [\omega]^2 \to 4$ defined by $g(x) = f_1(x) + 2 \cdot f_2(x)$ is computable and has the property that every set H_g homogeneous for g is both homogeneous for f_1 and homogeneous for f_2.

The following purely combinatorial proposition is a first step toward resolving Question 5.5.3

Proposition 5.5.5. *There exist (computable) $f_1\colon [\omega]^2 \to 2$ and $f_2\colon [\omega]^2 \to 2$ such there is no $g\colon [\omega]^2 \to 2$ with the property that every set H_g homogeneous for g is both homogeneous for f_1 and homogeneous for f_2.*

Proof. Let $\langle\cdot,\cdot\rangle\colon [\omega]^2 \to \omega$ be a fixed effective bijective coding of pairs of natural numbers. Define $f_1\colon [\omega]^2 \to 2$ and $f_2\colon [\omega]^2 \to 2$ by

$$f_1(\{\langle i,a\rangle,\langle j,b\rangle\}) = \begin{cases} 0 & \text{if } i = j \\ 1 & \text{otherwise} \end{cases}$$

$$f_2(\{\langle i,a\rangle,\langle j,b\rangle\}) = \begin{cases} 0 & \text{if } i \equiv j \pmod 2 \\ 1 & \text{otherwise} \end{cases}$$

For each $i \in \omega$, let $A_i = \{\langle i,a\rangle : a \in \omega\}$. Notice that an infinite set H is homogeneous for both f_1 and f_2 if and only if one of the following is true.

(1) There exists an i such that $H \subseteq A_i$.

(2) For all even i, we have $|H \cap A_i| \leq 1$, and for all odd i, we have $H \cap A_i = \emptyset$.

(3) For all odd i, we have $|H \cap A_i| \leq 1$, and for all even i, we have $H \cap A_i = \emptyset$.

Suppose that $g\colon [\omega]^2 \to 2$ is such that every set H homogeneous for g is homogeneous for both f_1 and f_2. For each $i \in \omega$, apply Ramsey's Theorem to $g \restriction [A_i]^2 \colon [A_i]^2 \to 2$ to obtain $B_i \subseteq A_i$ such that B_i is homogeneous for g. For each $i \in \omega$, let $q_i < 2$ be such that $g([B_i]^2) = \{q_i\}$.

Fix $i \in \omega$. Suppose that $j, b \in \omega$ with $i \neq j$ and $\langle j, b\rangle \in B_j$. If $Z = \{a \in \omega : \langle i, a\rangle \in B_i$ and $g(\{\langle i, a\rangle, \langle j, b\rangle\}) = q_i\}$ is infinite, then $H = \{\langle j, b\rangle\} \cup \{\langle i, a\rangle : a \in Z\}$ is a homogeneous set for g which does not satisfy any of (1)-(3) above (since $H \cap A_i$ is infinite and $|H \cap A_j| = 1$), a contradiction. Therefore, for all sufficiently large a with $\langle i, a\rangle \in B_i$, we have $g(\{\langle i, a\rangle, \langle j, b\rangle\}) = 1 - q_i$.

Suppose now that there exists $i \in \omega$ such that $Z = \{j \in \omega : q_j = 1 - q_i\}$ is infinite. Fix such an i and enumerate Z as $j_2 < j_3 < j_4 < \dots$. Fix b_0, b_1 with $b_0 \neq b_1$ such that $\langle i, b_0\rangle, \langle i, b_1\rangle \in B_i$, and let $j_0 = j_1 = i$. Suppose we have already defined $b_0, b_1, \dots, b_m$ with $m \geq 1$ and assume inductively that

- $\langle j_k, b_k\rangle \in B_{j_k}$ for all $k \leq m$.
- $g(\{\langle j_k, b_k\rangle, \langle j_l, b_l\rangle\}) = q_i$ for all k, l with $0 \leq k < l \leq m$.

We now define b_{m+1}. By the previous paragraph, for each $k \leq m$, all sufficiently large a with $\langle j_{m+1}, a\rangle \in B_{j_{m+1}}$ satisfy $g(\{\langle j_{m+1}, a\rangle, \langle j_k, b_k\rangle\}) = 1 - q_{j_{m+1}} = q_i$. Therefore, we may let b_{m+1} be the least a such that $\langle j_{m+1}, a\rangle \in B_{j_{m+1}}$ and $g(\{\langle j_{m+1}, a\rangle, \langle j_k, b_k\rangle\}) = q_i$ for all $k \leq m$. Then the inductive hypothesis holds and we may continue. However, the set $H = \{\langle j_m, b_m\rangle : m \in \omega\}$ is a homogeneous set for g which does not satisfy any of (1)-(3) above (since $|H \cap A_i| = 2$ and $|H \cap A_{j_m}| = 1$ for all $m \in \omega$), a contradiction.

Since at least one of the sets $\{j \in \omega : q_j = 0\}$ and $\{j \in \omega : q_j = 1\}$ is infinite, it follows from the previous paragraph that $q_i = q_j$ for all $i, j \in \omega$. Let $q = q_i$ for some (all) $i \in \omega$. Fix b_0 such that $\langle 0, b_0\rangle \in B_0$. Suppose we have already defined $b_0, b_1, \dots, b_m$ with $m \geq 0$ and assume inductively that

- $\langle k, b_k\rangle \in B_k$ for all $k \leq m$.
- $g(\{\langle k, b_k\rangle, \langle l, b_l\rangle\}) = 1 - q$ for all k, l with $0 \leq k < l \leq m$.

We now define b_{m+1}. As above, for each $k \leq m$, all sufficiently large a with $\langle m+1, a\rangle \in B_{m+1}$ satisfy $g(\{\langle m+1, a\rangle, \langle k, b_k\rangle\}) = 1 - q$. Therefore, we may let b_{m+1} be the least a such that $\langle m+1, a\rangle \in B_{m+1}$ and $g(\{\langle m+1, a\rangle, \langle k, b_k\rangle\}) = 1 - q$ for all $k \leq m$. Then the inductive hypothesis holds and we may continue. However, the set $H = \{\langle m, b_m\rangle : m \in \omega\}$ is a homogeneous set for g which does not satisfy any of (1)-(3) above (since $|H \cap A_j| = 1$ for all $j \in \omega$), a contradiction. $\square$

Chapter 6

Generalized Cohesive Sets

6.1 Notions of Cohesiveness

Recall the definition of r-cohesive sets in Chapter 2 and the equivalent characterization given by Claim 2.2.9.

Claim 6.1.1. *A set V is r-cohesive if and only if it is infinite, and for every computable $f\colon [\omega]^1 \to 2$, there exists a finite $z \subseteq \omega$ such that $V \backslash z$ is homogeneous for f.*

With this characterization in mind, it is possible to extend the definition of cohesiveness to higher exponents.

Definition 6.1.2 (Hummel and Jockusch [7, Definition 1.3]). Fix $n \geq 1$. A set V is *n-r-cohesive* if V is infinite, and for every computable $f\colon [\omega]^n \to 2$, there exists a finite set z such that $V \backslash z$ is homogeneous for f.

Similarly, we can define notions of cohesiveness for the Canonical Ramsey Theorem and the Regressive Function Theorem.

Definition 6.1.3. Fix $n \geq 1$.

- A set V is *n-c-cohesive* if V is infinite, and for every computable $f\colon [\omega]^n \to \omega$, there exists a finite set z such that $V \backslash z$ is canonical for f.
- A set V is *strongly n-c-cohesive* if V is infinite, and for every $p \geq 1$ and every computable $f\colon [\omega]^n \to \omega \times p$, there exists a finite set z such that $V \backslash z$ is canonical for f.
- A set V is *n-g-cohesive* if V is infinite, and for every computable regressive $f\colon [\omega]^n \to \omega$, there exists a finite set z such that $V \backslash z$ is minhomogeneous for f.

Remark 6.1.4. By Claim 2.3.3 and Claim 2.4.4, every n-c-cohesive set is both n-r-cohesive and n-g-cohesive.

To inductively construct n-c-cohesive sets, it seems that we need to construct strongly n-c-cohesive sets. Fortunately, we have the following simple result.

Claim 6.1.5. *For any $n \geq 1$, a set V is n-c-cohesive if and only if it is strongly n-c-cohesive.*

Proof. Clearly, if V is strongly n-c-cohesive then V is n-c-cohesive. Suppose that V is n-c-cohesive, and let $f\colon [\omega]^n \to \omega \times p$ be computable. Since $\pi_1 \circ f\colon [\omega]^n \to \omega$ and $\pi_2 \circ f\colon [\omega]^n \to p$ are computable, there exists a finite set z_1 such that $V \backslash z_1$ is canonical for $\pi_1 \circ f$ and there exists a finite set z_2 such that $V \backslash z_2$ is canonical for $\pi_2 \circ f$. By Claim 2.3.3, $V \backslash z_2$ is homogeneous for $\pi_2 \circ f$. Therefore, $V \backslash (z_1 \cup z_2)$ is canonical for f. $\square$

6.2 1-c-cohesive Sets

Madan and Robinson [19] studied 1-c-cohesive sets, which they called 1-1 sets (notice that an infinite set A is 1-c-cohesive if and if only if every computable $f\colon \omega \to \omega$ is either eventually constant on A or eventually 1-1 on A). Their paper (see [19, Theorem 1.2]) contains the following fact.

Theorem 6.2.1. *Every 1-c-cohesive set is dense immune, and hence of high degree.*

The following result also appears in [19, Theorem 3.1], and is originally due to Owings [20].

Theorem 6.2.2. *If P is co-maximal, then P is 1-c-cohesive.*

Since maximal sets appear in every high c.e. degree, it follows that every high c.e. degree contains a 1-c-cohesive set. We now generalize this result.

Theorem 6.2.3. *If $\mathbf{a}$ is high (i.e. $\mathbf{a}' \geq \mathbf{0}''$) and B is infinite and computable, then $\mathbf{a}$ contains a 1-c-cohesive subset of B.*

Proof. Suppose that $\mathbf{a}' \geq \mathbf{0}''$. We first show that there is an $\mathbf{a}$-computable 1-c-cohesive subset of B. Fix e_0 such that $B = \varphi_{e_0}$. Using a $0''$-oracle, we inductively define a function $g\colon \omega \to \omega$ as follows. Let $g(0) = e_0$. Suppose that we have defined $g(k)$ for some $k \geq 0$ and that $\varphi_{g(k)}$ is total, $\{0, 1\}$-valued, and infinite. If φ_k is not total (a question we can answer using a $0''$-oracle), let $g(k+1) = g(k)$. Suppose that φ_k is total. Using a $0''$-oracle again, determine whether φ_k is unbounded on $\varphi_{g(k)}$, i.e. whether $(\forall m)(\exists b)[\varphi_{g(k)}(b) = 1 \wedge \varphi_k(b) \geq m]$.

If φ_k is unbounded on $\varphi_{g(k)}$, use a $0''$-oracle to let $g(k+1)$ be the least $e \in \omega$ such that:

- φ_e is total, $\{0, 1\}$-valued, and infinite.
- $\varphi_{g(k)}(b) = 1$ for all b with $\varphi_e(b) = 1$.

- $\varphi_k(a) \neq \varphi_k(b)$ for all $a, b \in \omega$ with $a \neq b$ and $\varphi_e(a) = 1 = \varphi_e(b)$.

To see that such an e exists, inductively define a strictly increasing computable function p as follows. Let $p(0)$ be the least b such that $\varphi_{g(k)}(b) = 1$. Suppose that we have defined $p(i)$ for all $i \leq l$. There exists b with $\varphi_{g(k)}(b) = 1$ such that $b > p(i)$ and $\varphi_k(b) \neq \varphi_k(p(i))$ for all $i \leq l$ since φ_k is unbounded on $\varphi_{g(k)}$. Let $p(l+1)$ be the least such b, which can be found effectively. Since p is computable and strictly increasing, there is an e such that $\varphi_e = \mathrm{range}(p)$, and this e satisfies the above.

If φ_k is bounded on $\varphi_{g(k)}$, proceed as follows. There exists c such that the set $Z_c = \{b \in \omega : \varphi_{g(k)}(b) = 1$ and $\varphi_k(b) = c\}$ is infinite. Fix the least such c using a $0''$-oracle. Since Z_c is computable, there exists e such that $\varphi_e = Z_c$, and we let $g(k+1)$ be the least such e (which we can again find using a $0''$-oracle).

Notice that

- g is $0''$-computable.
- $\varphi_{g(k)}$ is total, $\{0,1\}$-valued, and infinite for all $k \in \omega$.
- $B = \varphi_{g(0)} \supseteq \varphi_{g(1)} \supseteq \varphi_{g(2)} \supseteq \cdots$.
- For all $k \in \omega$ with φ_k total, $\varphi_{g(k)}$ is canonical for φ_k.

Since $\deg(g) \leq \mathbf{0}''$ and $\mathbf{a}' \geq \mathbf{0}''$, there exists an $\mathbf{a}$-computable $g_1\colon \omega^2 \to \omega$ such that $g(k) = \lim_s g_1(k,s)$ for all $k \in \omega$. We now define an 1-c-cohesive set V with $V \subseteq B$ by inductively defining its principal function p_V using an $\mathbf{a}$-oracle. Let $p_V(0) = \min(B)$. Suppose that $p_V(m)$ is defined. Let s_{m+1} be the least s such that

$$(\exists b)[p_V(m) < b \leq s \wedge b \in B \wedge (\forall k \leq m)[\varphi_{g_1(k,s),s}(b) \downarrow = 1]]$$

Notice that such an s exists because for all sufficiently large t, we have $g_1(k,t) = g(k)$ for all $k \leq m$. Let $p_V(m+1)$ be the least b satisfying the above with $s = s_{m+1}$.

We now show that V is a 1-c-cohesive subset of B. Clearly, $V \subseteq B$ by construction. Suppose that φ_k is total. Fix t such that $g_1(i,s) = g(i)$ for all $i \leq k$ and all $s \geq t$. By construction, for all $m \geq \max\{k,t\}$, we have $\varphi_{g(k)}(p_V(m)) = 1$. Therefore, letting $z = \{p_V(i) : i < \max\{k,t\}\}$, we see that $V \backslash z \subseteq \varphi_{g(k)}$. Since $\varphi_{g(k)}$ is canonical for φ_k, it follows that $V \backslash z$ is canonical for φ_k.

Therefore, there is an $\mathbf{a}$-computable 1-c-cohesive subset of B. Since there is an arithmetical 1-c-cohesive subset of B (from the above), and any infinite subset of a 1-c-cohesive subset of B is a 1-c-cohesive subset of B, it follows from [13, Theorem 1] that the degrees of 1-c-cohesive sets are closed upwards. Hence, there is a 1-c-cohesive subset of B of degree $\mathbf{a}$. ☐

6.3 n-c-cohesive Sets

We can lift results about 1-c-cohesive sets to higher exponents by uniformly building a set A and a family of functions g_k which simultaneously code (modulo finite sets) precanonical pairs for all computable $f\colon [\omega]^{n+1} \to \omega$, and using a relativized n-c-cohesive set.

Theorem 6.3.1. *Suppose that $X \subseteq \omega$, $n \geq 1$, $B \subseteq \omega$ is infinite and X-computable, and $\mathbf{a}$ is a Turing degree which is high relative to $deg(X)^{(2n-2)}$ (i.e. $\mathbf{a} \geq deg(X)^{(2n-2)}$ and $\mathbf{a}' \geq deg(X)^{(2n)}$). There exists an $\mathbf{a}$-computable set $V \subseteq B$ which is n-c-cohesive relative to X.*

Proof. The proof is by induction on n. The case $n = 1$ follows by relativizing Theorem 6.2.3. Suppose that the result holds for n. Let $X \subseteq \omega$, $B \subseteq \omega$ be infinite and X-computable, and let $\mathbf{a}$ be a Turing degree which is high relative to $\deg(X)^{(2n)}$. Since $\{e \in \omega : \varphi_e^X \text{ is total}\} \leq_T X''$, the function $f\colon \omega \to \omega$ defined recursively by

$$f(0) = (\mu e)[\varphi_e^X \text{ is total}]$$
$$f(k+1) = (\mu e)[e > f(k) \text{ and } \varphi_e^X \text{ is total}]$$

is X''-computable. Using an X''-oracle, we uniformly build a set $A \subseteq B$ with $A = \{a_0 < a_1 < a_2 < \dots\}$, and functions g_k for each $k \in \omega$. For each $k \in \omega$, if we let $A_k = \{a_i : i \geq k\}$, then $g_k\colon [A_k]^n \to \omega \times 2$, and (A_k, g_k) will be a precanonical pair for $\varphi_{f(k)}^X$ (viewing $\varphi_{f(k)}^X$ as a function from $[\omega]^{n+1}$ to ω). The proof is completely analogous to the proof of Proposition 3.2.3 relative to X, except that in the construction of A, once we define a_{m+1}, we proceed in order for each $k \leq m+1$ to thin out the set constructed according to each of the X-computable functions $\varphi_{f(k)}^X$ and define g_k on all elements of $[\{a_i : k \leq i \leq m+1\}]^n$ whose last element is a_{m+1}.

By the inductive hypothesis relative to X'' and Claim 6.1.5 relative to X'', there is a strongly n-c-cohesive set V relative to X'' such that $V \subseteq A$ and $\deg(V) \leq \mathbf{a}$ (since $\mathbf{a}$ is high relative to $\deg(X)^{(2n)} = \deg(X'')^{(2n-2)}$). Suppose that $h\colon [\omega]^{n+1} \to \omega$ is X-computable, and fix $k \in \omega$ with $h = \varphi_{f(k)}^X$. Then $g_k\colon [A_k]^n \to \omega \times 2$ is X''-computable, so there is a finite set z such that $V \backslash z$ is canonical for g_k. By Claim 3.1.5, $V \backslash z$ is canonical for f. Therefore, V is an $\mathbf{a}$-computable $(n+1)$-c-cohesive subset of B. □

Corollary 6.3.2. *Suppose that $n \geq 1$, $B \subseteq \omega$ is infinite and computable, and $\mathbf{a}$ is a Turing degree which is high relative to $\mathbf{0}^{(2n-2)}$. There exists an n-c-cohesive set of degree $\mathbf{a}$.*

Proof. Since there is an arithmetical n-c-cohesive subset of B (from Theorem 6.3.1), and any infinite subset of a n-c-cohesive subset of B is also an n-c-cohesive subset of B, it follows from [13, Theorem 1] that the degrees of n-c-cohesive subsets of B are closed upwards. Therefore, by Theorem 6.3.1, there is an n-c-cohesive subset of B of degree $\mathbf{a}$. $\square$

6.4 n-g-cohesive Sets

Notice that given any regressive $f\colon [\omega]^1 \to \omega$, every infinite set is trivially minhomogeneous for f. Therefore, every infinite set is 1-g-cohesive relative to every $X \subseteq \omega$.

Again, we can lift results about 1-g-cohesive sets to higher exponents by uniformly building a set A and a family of functions g_k which simultaneously code (modulo finite sets) preminhomogeneous pairs for all computable regressive $f\colon [\omega]^{n+1} \to \omega$, and using a relativized n-g-cohesive set.

Theorem 6.4.1. *Suppose that $X \subseteq \omega$, $n \geq 1$, $B \subseteq \omega$ is infinite and X-computable, and $\mathbf{a}$ is a Turing degree such that $\mathbf{a} \gg deg(X)^{(n-1)}$. There exists an $\mathbf{a}$-computable set $V \subseteq B$ which is n-g-cohesive relative to X.*

Proof. The proof is by induction on n. The case $n = 1$ is trivial from the above. Suppose that the result holds for n. Let $X \subseteq \omega$, $B \subseteq \omega$ be infinite and X-computable and let $\mathbf{a}$ be a Turing degree such that $\mathbf{a} \gg \deg(X)^{(n)}$. By the relativized Low Basis Theorem, we may fix $\mathbf{b} \gg \deg(X)$ with $\mathbf{b}' \leq \deg(X)'$ and $\mathbf{c} \gg \deg(X)'$ with $\mathbf{c}' \leq \deg(X)''$. By Proposition 5.4.4 relativized to X, the class of X-computable regressive functions from $[\omega]^{n+1}$ to ω is $\mathbf{b}$-subuniform, so there exists a $\mathbf{b}$-computable $f\colon \omega \times [\omega]^{n+1} \to \omega$ such that the function $y \mapsto f(k, y)$ is regressive for all k, and for all X-computable regressive $g\colon [\omega]^{n+1} \to \omega$, there exists $k \in \omega$ such that $g(y) = f(k, y)$ for all $y \in [\omega]^{n+1}$.

For each $k \in \omega$, let $f_k\colon [\omega]^{n+1} \to \omega$ be such that $f_k(y) = f(k, y)$ for all $y \in [\omega]^{n+1}$. Using the fact that $\mathbf{c} \gg \deg(X)' \geq \mathbf{b}'$, we use a $\mathbf{c}$-oracle to uniformly build a set $A \subseteq B$ with $A = \{a_0 < a_1 < a_2 < \dots\}$, and functions g_k for each $k \in \omega$. For each $k \in \omega$, if we let $A_k = \{a_i : i \geq k\}$, then $g_k\colon [A_k]^n \to \omega$, and (A_k, g_k) will be a preminhomogeneous pair for f_k. The proof is completely analogous to the proof of Proposition 4.2.3 relative to $\mathbf{b}$, except that in the construction of A, to determine a_{m+1}, we consider the values $f_k(x, b)$ for all $x \in [\{a_i : k \leq i \leq m\}]^n$ and define g_k on all elements of $[\{a_i : k \leq i \leq m + 1\}]^n$ whose last element is a_{m+1}, for each $k \leq m$.

By the inductive hypothesis relative to $\mathbf{c}$, there is an n-g-cohesive set V relative to $\mathbf{c}$ such that $V \subseteq A$ and $\deg(V) \leq \mathbf{a}$ (since $\mathbf{a} \gg \deg(X)^{(n)} = \deg(X')^{(n-1)} \geq \mathbf{c}^{(n-1)}$). Suppose that $h\colon [\omega]^{n+1} \to \omega$ is X-computable and regressive, and fix $k \in \omega$ with $h = f_k$. Then $g_k\colon [A_k]^n \to \omega$ is $\mathbf{c}$-computable, so there is a finite set z such that $V \backslash z$ is minhomogeneous for g_k. By Claim 4.2.2, $V \backslash z$ is minhomogeneous for f. Therefore, V is an $\mathbf{a}$-computable $(n + 1)$-g-cohesive subset of B. $\square$

Corollary 6.4.2. *Suppose that $n \geq 1$, $B \subseteq \omega$ is infinite and computable, and* $\mathbf{a}$ *is a Turing degree such that* $\mathbf{a} \gg \mathbf{0}^{(n-1)}$*. There exists an n-g-cohesive set of degree* $\mathbf{a}$*.*

Proof. Since there is an arithmetical n-g-cohesive subset of B (from Theorem 6.4.1), and any infinite subset of an n-g-cohesive subset of B is also an n-g-cohesive subset of B, it follows from [13, Theorem 1] that the degrees of n-g-cohesive subsets of B are closed upwards. Therefore, by Theorem 6.4.1, there is an n-g-cohesive subset of B of degree $\mathbf{a}$. □

Remark 6.4.3. By Theorem 4.3.2, given $n \geq 2$, every n-g-cohesive set V satisfies $\deg(V) \gg \mathbf{0}^{(n-1)}$. Therefore, for $n \geq 2$, Theorem 6.4.1 gives a sharp characterization of the degrees of n-g-cohesive sets.

Chapter 7

Reverse Mathematics

7.1 The Reverse Mathematics of Partition Theorems

In this section, we obtain some basic reverse mathematics corollaries from the computability-theoretic analysis we've carried out thus far.

Definition 7.1.1. The following definitions are made in second-order arithmetic.

(1) RT^n_p is the statement that every $f\colon [\mathbb{N}]^n \to p$ has a homogeneous set.

(2) RT^n is the statement that for all $p \geq 1$, every $f\colon [\mathbb{N}]^n \to p$ has a homogeneous set.

(3) RT is the statement that for all $n, p \geq 1$, every $f\colon [\mathbb{N}]^n \to p$ has a homogeneous set.

(4) CRT^n is the statement that every $f\colon [\mathbb{N}]^n \to \mathbb{N}$ has a canonical set.

(5) CRT is the statement that for all $n \geq 1$, every $f\colon [\mathbb{N}]^n \to \mathbb{N}$ has a canonical set.

(6) REG^n is the statement that every regressive $f\colon [\mathbb{N}]^n \to \mathbb{N}$ has a minhomogeneous set.

(7) REG is the statement that for all $n \geq 1$, every regressive $f\colon [\mathbb{N}]^n \to \mathbb{N}$ has a minhomogeneous set.

(8) ACA_0' is the statement that for all sets Z and all n, the n^{th} jump of Z exists.

(9) $\mathsf{B}\Gamma$ (where Γ is a set of formulas) is the statement of Γ-bounding, i.e. for any formula $\theta(a,b) \in \Gamma$ we have

$$(\forall c)[(\forall a < c)(\exists b)\theta(a,b) \to (\exists m)(\forall a < c)(\exists b < m)\theta(a,b)]$$

Proposition 7.1.2. *The following are equivalent over* RCA_0

(1) ACA_0

(2) CRT^n *for any fixed* $n \geq 2$.

(3) REG^n *for any fixed* $n \geq 2$.

(4) RT^n *for any fixed* $n \geq 3$.

(5) RT^n_p *for any fixed* $n \geq 3$ *and* $p \geq 2$.

Proof. To see that (1) implies (2), examine the proof of Theorem 3.1.3 and notice that it can be formalized (in a completely straightforward manner) in ACA_0. Since the proofs of Claim 2.4.4 and Claim 2.3.3 can be carried out in RCA_0, it follows that (2) implies (3) and (4). Formalizing the proof of Theorem 4.3.1 in RCA_0 gives (3) implies (1). Clearly, (4) implies (5), and formalizing the proof of Proposition 2.2.18 in RCA_0 gives (5) implies (1). $\square$

Remark 7.1.3. At the end of [17], Kanamori and McAloon state that the implication $\mathsf{REG}^2 \to \mathsf{ACA}_0$ over RCA_0 is due to Clote. Hirst (see [6, Theorem 6.14]), in his thesis, proved that the stronger statement "Every h-regressive $f\colon [\mathbb{N}]^2 \to \mathbb{N}$ has a minhomogeneous set" implies ACA_0 over RCA_0.

Proposition 7.1.4. *The following are equivalent over* RCA_0*:*

(1) ACA_0'

(2) CRT

(3) REG

(4) RT

Proof. To see that (1) implies (2), examine the proof of Theorem 3.1.3 and notice that it can be formalized for all exponents n (in a completely straightforward manner) in ACA_0'. Since the proofs of Claim 2.4.4 and Claim 2.3.3 can be carried out in RCA_0, it follows that (2) implies (3) and (4). Formalizing the proof of Theorem 4.3.2 in RCA_0 gives (3) implies (1), and formalizing the proof of Proposition 2.2.18 in RCA_0 gives (4) implies (1). $\square$

Proposition 7.1.5. *The following are equivalent over* RCA_0*:*

(1) $\mathsf{B}\Pi^0_1$

(2) $\mathsf{B}\Sigma^0_2$

(3) RT^1

(4) CRT^1

Proof. The equivalence of (1) and (2) is standard and can be found in [5, Lemma 2.10]. The equivalence of (1) and (3) is due to Hirst [6, Theorem 6.4], and can also be found in [1, Theorem 2.10]. Since the proof of Claim 2.3.3 can be carried out in RCA_0, it follows that (4) implies (3).

We now show that (3) implies (4). Let $\mathfrak{M}$ be a model of $\mathsf{RCA}_0 + \mathsf{RT}^1$ and let $\mathbb{N}$ be the set of natural numbers in $\mathfrak{M}$. Suppose that $f\colon \mathbb{N} \to \mathbb{N}$ and $f \in \mathfrak{M}$. If there exists $p \in \mathbb{N}$ such that $f(n) \leq p$ for all $n \in \mathbb{N}$, then there exists a set $H \in \mathfrak{M}$ which is homogeneous for f since RT^1_{p+1} holds in $\mathfrak{M}$, and such an H is canonical for f. Suppose then that the range of f is unbounded, i.e. for every $p \in \mathbb{N}$, there exists an $n \in \mathbb{N}$ with $f(n) > p$. Since $\mathfrak{M}$ satisfies Δ^0_1 comprehension, we may recursively define a function $g \in \mathfrak{M}$ as follows. Let $g(0) = 0$, and given $g(n)$, let $g(n+1)$ be the least $k \in \mathbb{N}$ such that $k > g(n)$ and $f(k) > f(g(n))$. Since g is strictly increasing, and $g \in \mathfrak{M}$, it follows that range(g) is infinite and range$(g) \in \mathfrak{M}$. Notice that range(g) is canonical for f. □

7.2 Open Questions

Many open questions about the reverse mathematical strength of RT^2_2 (and simple corollaries of it) remain. The following is perhaps the most fundamental.

Question 7.2.1 (Seetapun). *Does* RT^2_2 *imply* WKL_0 *over* RCA_0*?*

Definition 7.2.2. The following definitions are made in second-order arithmetic.

(1) SRT^2_2 is the statement that every stable $f\colon [\mathbb{N}]^2 \to 2$ has a homogeneous set.

(2) CAC is the statement that every infinite partially ordered set has either an infinite chain or an infinite antichain.

Question 7.2.3. *Does* SRT^2_2 *imply* RT^2_2 *over* RCA_0*?*

Question 7.2.4. *Does* CAC *imply* RT^2_2 *over* RCA_0*?*

Appendix A

A Corrected Proof

A.1 $\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \to \mathsf{COH}$

Definition A.1.1. In second-order arithmetic, let COH be the statement that for any sequence of sets $(R_i)_{i\in\mathbb{N}}$, there is an infinite set A such that for all $i \in \mathbb{N}$, either $A \subseteq^* R_i$ or $A \subseteq^* \overline{R_i}$.

Definition A.1.2. In second-order arithmetic, let SRT^2_2 be the statement that every stable $f\colon [\mathbb{N}]^2 \to 2$ has a homogeneous set.

Cholak, Jockusch, and Slaman (see [1, Lemma 7.11]) claimed that $\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \leftrightarrow (\mathsf{SRT}^2_2 + \mathsf{COH})$, but their proof of $\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \to \mathsf{COH}$ is seriously flawed. We now give a correct proof of this fact (Jockusch and Lempp [14] have independently discovered a similar proof).

Claim A.1.3. $\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \to \mathsf{COH}$.

Proof. The proof is similar to the proof of [1, Theorem 12.5], but we need a slightly more refined argument to show that the proof goes through in RCA_0 (the proof easily goes through in $\mathsf{RCA}_0 + \Sigma^0_2\text{-}\mathsf{IND}$).

Let $\mathfrak{M}$ be a model of $\mathsf{RCA}_0 + \mathsf{RT}^2_2$ and let $\mathbb{N}$ be the set of natural numbers in $\mathfrak{M}$. Let $(R_i)_{i\in\mathbb{N}} \in \mathfrak{M}$ be a sequence of sets. By including additional sets in the sequence $(R_i)_{i\in\mathbb{N}}$, if necessary, we may assume that for all $a, b \in \mathbb{N}$ with $a < b$, there exists $i \in \mathbb{N}$ such that $R_i(a) \neq R_i(b)$. Define $d\colon [\mathbb{N}]^2 \to \mathbb{N}$ by letting $d(a, b)$ be the least $i \in \mathbb{N}$ such that $R_i(a) \neq R_i(b)$. Define $f\colon [\mathbb{N}]^2 \to 2$ by letting $f(a, b) = R_{d(a,b)}(a)$. Notice that $d, f \in \mathfrak{M}$ since $\mathfrak{M} \models \mathsf{RCA}_0$.

Since $\mathfrak{M} \models \mathsf{RT}^2_2$, there exists $A \in \mathfrak{M}$ which is homogeneous for f. Let $(a_n)_{n\in\mathbb{N}}$ enumerate A in increasing order. Suppose first that $f([A]^2) = \{0\}$. We prove by induction on $i \in \mathbb{N}$ that for all finite sets $X \subseteq \mathbb{N}$, if $R_i(a_n) = 1$ and $R_i(a_{n+1}) = 0$ for all $n \in X$, then $|X| \leq 2^i - 1$. Notice that for any $n \in \mathbb{N}$, if $R_0(a_n) = 1$ and $R_0(a_{n+1}) = 0$, then $d(a_n, a_{n+1}) = 0$ and so $f(a_n, a_{n+1}) = 1$, contrary to the fact that $f([A]^2) = \{0\}$.

Therefore, if $X \subseteq \mathbb{N}$ is a finite set such that $R_0(a_n) = 1$ and $R_0(a_{n+1}) = 0$ for all $n \in X$, then $|X| = 0 = 2^0 - 1$, and the result holds for $i = 0$.

Suppose that $i \in \mathbb{N}$, $i > 0$, and the result holds for all $j < i$. Notice that, by the inductive hypothesis, for all finite sets $Y \subseteq \mathbb{N}$ and all $j < i$, if $R_j(a_n) = 0$ and $R_j(a_{n+1}) = 1$ for all $n \in Y$, then $|Y| \leq 2^j$. Let $X \subseteq \mathbb{N}$ be a finite set such that $R_i(a_n) = 1$ and $R_i(a_{n+1}) = 0$ for all $n \in X$. Since $f([A]^2) = \{0\}$, it must be the case that $d(a_n, a_{n+1}) < i$ for all $n \in X$, and moreover $R_{d(a_n,a_{n+1})}(a_n) = 0$ and $R_{d(a_n,a_{n+1})}(a_{n+1}) = 1$. For each $j < i$, let $Y_j = \{n \in X : d(a_n, a_{n+1}) = j\}$. Then $X = \bigcup_{j=0}^{i-1} Y_j$, hence $|X| \leq \sum_{j=0}^{i-1} |Y_j| \leq \sum_{j=0}^{i-1} 2^j = 2^i - 1$. Thus, the result holds for i.

Since the inductive statement is Π^0_1, and RCA_0 proves order induction for Π^0_1 statements, it follows that for all $i \in \mathbb{N}$ and all finite sets $X \subseteq \mathbb{N}$, if $R_i(a_n) = 1$ and $R_i(a_{n+1}) = 0$ for all $n \in X$, then $|X| \leq 2^i - 1$. Similarly, if $f([A]^2) = \{1\}$, then by interchanging the roles of 0 and 1 in the above arguments, we see that for all $i \in \mathbb{N}$ and all finite sets $X \subseteq \mathbb{N}$, if $R_i(a_n) = 0$ and $R_i(a_{n+1}) = 1$ for all $n \in X$, then $|X| \leq 2^i - 1$. Therefore, for all $i \in \mathbb{N}$, either $A \subseteq^* R_i$ or $A \subseteq^* \overline{R_i}$. It follows that $\mathfrak{M} \models \mathsf{COH}$. $\square$

Corollary A.1.4. *$\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \leftrightarrow (\mathsf{SRT}^2_2 + \mathsf{COH})$.*

Proof. Clearly, $\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \to \mathsf{SRT}^2_2$ and Claim A.1.3 gives $\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \to \mathsf{COH}$. To see that $\mathsf{RCA}_0 \vdash (\mathsf{SRT}^2_2 + \mathsf{COH}) \to \mathsf{RT}^2_2$, let $\mathfrak{M}$ be a model of $\mathsf{RCA}_0 + \mathsf{SRT}^2_2 + \mathsf{COH}$ and let $\mathbb{N}$ be the set of natural numbers in $\mathfrak{M}$. Let $f\colon [\mathbb{N}]^2 \to 2$ with $f \in \mathfrak{M}$. For each $i \in \mathbb{N}$, let $R_i = \{j > i : f(i,j) = 0\}$, and notice that $(R_i)_{i \in \mathbb{N}} \in \mathfrak{M}$. Since $\mathfrak{M} \models \mathsf{COH}$, there exists an infinite $A \in \mathfrak{M}$ such that for all $i \in \mathbb{N}$, either $A \subseteq^* R_i$ or $A \subseteq^* \overline{R_i}$. Let $(a_n)_{n \in \mathbb{N}}$ enumerate A in increasing order. Define $g\colon [\mathbb{N}]^2 \to 2$ by letting $g(n,m) = f(a_n, a_m)$ and notice that $g \in \mathfrak{M}$ and g is stable. Since $\mathfrak{M} \models \mathsf{SRT}^2_2$, there exists $H \in \mathfrak{M}$ homogeneous for g. We then have $g(H) \in \mathfrak{M}$ and $g(H)$ is homogeneous for f. It follows that $\mathfrak{M} \models \mathsf{RT}^2_2$. $\square$

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Vita

Joseph Roy Mileti was born on July 1, 1978 in Parma, Ohio. He received his B.S. in Computer Science and Discrete Mathematics/Logic with honors at Carnegie Mellon University in 1999. At the University of Illinois at Urbana-Champaign, he was supported for many years through VIGRE and GAANN fellowships and received the 2003 Mathematics Instructional Award from the Mathematics Department. He will be an L.E. Dickson Instructor at the University of Chicago beginning September 1, 2004.