Midterm Exam: Due Friday, March 17 at 5:00pm

- You are free to use the course textbook, the homework solutions, your personal notes, and your previous homework in solving these problems. However, you may *not* consult any other sources (other books, online resources, etc.).
- Aside from questions that you directly ask me, you may *not* communicate with anybody else (student or otherwise) about the problems on the exam.

Problem 1: (6 points) Let $a, b \in \mathbb{R}$ with $a < b$. Let $A \subseteq (a, b)$ with $\lambda^*(A) = 0$. Let $B = (a, b)\backslash A$.
a. Show that B is uncountable.
b. Show that B is dense in (a, b), i.e. for all $c, d \in \mathbb{R}$ with $a < c < d < b$, we have $(c, d) \cap B \neq \emptyset$.

Problem 2: (6 points) Let $(X, \mathcal{S}, \mu)$ be a measure space. Show that if $A, B \in \mathcal{S}$, then

$$\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B).$$

Keep in mind that some of these values might be infinite, so be sure that your argument handles that situation in some way.

Problem 3: (6 points) Let $\mathcal{S} \subseteq \mathcal{P}(\mathbb{R})$ be the σ-algebra generated by the open and the null sets, i.e. $\mathcal{S}$ is the smallest σ-algebra that contains every open set and every null set. Show that $\mathcal{S}$ is the collection of measurable sets.

Problem 4: (6 points) Let $x \in [0, 1)$, and suppose that x is not a dyadic rational. Either prove or find a counterexample: If x is a periodic point of the doubling map, then x is a periodic point of the tent map.
Note: Recall that x is a period point of $T\colon X \to X$ if there exists $n \in \mathbb{N}^+$ with $T^n(x) = x$.

Problem 5: (8 points) Let $A \subseteq \mathbb{R}$ be a bounded set. Show that A is measurable if and only if for every $\varepsilon > 0$, there exists a compact set $K \subseteq A$ with $\lambda^*(K) > \lambda^*(A) - \varepsilon$.

Problem 6: (9 points) Let $A, B \subseteq \mathbb{R}$. Recall that $A + B = \{a + b : a \in A, b \in B\}$.
a. Show that if A and B are both open, then $A + B$ is open.
b. Show that if A is closed and B is compact, then $A + B$ is closed.
c. Show that if A and B are both closed, then $A + B$ is measurable.

Bonus: (2 points) Give an example of two closed sets $A, B \subseteq \mathbb{R}$ such that $A + B$ is not closed.