Homework 9: Due Friday, April 28

Problem 1:
a. Let $(X, \mathcal{S}, \mu)$ be a measure space, and let $f\colon X \to \mathbb{R}$ be an integrable function. Show that for every $\varepsilon > 0$, there exists a simple function $\varphi\colon X \to \mathbb{R}$ such that

$$\int |f - \varphi| \; d\mu < \varepsilon.$$

b. Consider the special case where $(X, \mathcal{S}, \mu) = (\mathbb{R}, \mathcal{M}, \lambda)$. Let $f\colon \mathbb{R} \to \mathbb{R}$ be an integrable function. Show that for every $\varepsilon > 0$, there exists a step function $g\colon X \to \mathbb{R}$ such that

$$\int |f - g| \; d\lambda < \varepsilon.$$

Problem 2: Let $(X, \mathcal{S}, \mu)$ be a measure space, and let $f\colon X \to \mathbb{R}$ be a bounded measurable function. Recall that we originally thought about defining $||f||_\infty$ to be $\sup\{|f(x)| : x \in X\}$. However, unlike our definition for L^p, such a definition would not be stable if we changed the function on a set of measure 0. As a result, we actually define

$$\begin{aligned} ||f||_\infty &= \inf\{c \in \mathbb{R} : \mu(\{x \in X : |f(x)| > c\}) = 0\} \\ &= \inf\{c \in \mathbb{R}^+ : \mu(f^{-1}((-\infty, -c) \cup (c, \infty))) = 0\}. \end{aligned}$$

The quantity on the right is sometimes called the *essential supremum* of f.
a. Show that if $f, g\colon X \to \mathbb{R}$ are both bounded measurable functions, and if $f = g$ a.e., then $||f||_\infty = ||g||_\infty$.
b. Show that if $f, g\colon X \to \mathbb{R}$ are both bounded measurable functions, then

$$||f + g||_\infty \leq ||f||_\infty + ||g||_\infty.$$

Problem 3: Let $(X, \mathcal{S}, \mu)$ be a measure space. Show that if $f\colon X \to \mathbb{R}$ is integrable, and if $g\colon X \to \mathbb{R}$ is measurable and bounded, then $f \cdot g$ is integrable, and

$$\int |fg| \; d\mu \leq ||f||_1 \cdot ||g||_\infty.$$

In other words, if $f \in L^1$ and $g \in L^\infty$, then $f \cdot g \in L^1$, and the above inequality is true.

Problem 4: Recall that we defined an inner product on L^2 by letting $\langle f, g\rangle = \int fg \; d\mu$. Consider the functions $f(x) = 1$, $g_n(x) = \cos(nx)$, and $h_n(x) = \sin(nx)$ defined on the interval $[0, 2\pi]$. Show that any two distinct elements of $\{f, g_1, g_2, \dots, h_1, h_2, \dots\}$ are orthogonal on the interval $[0, 2\pi]$ under this inner product, i.e. show that the integral over $[0, 2\pi]$ of the product of any two distinct elements of this set equals 0.
Hint: Some trigonometric identities are useful.

Problem 5: Let $f\colon \mathbb{R} \to \mathbb{R}$ be an integrable function. For each $n \in \mathbb{N}^+$, let $h_n\colon \mathbb{R} \to \mathbb{R}$ be the function $h_n(x) = \cos(nx)$. Show that

$$\lim_{n \to \infty} \int f \cdot h_n \; d\lambda = 0.$$

Hint: First consider the case where f is a step function.
Aside: This result is known as the Riemann-Lebesgue Lemma.