Homework 8: Due Friday, April 21

Problem 1: (from Exercise 4.2.7) Show that if $g\colon \mathbb{R} \to \mathbb{R}$ is measurable and $f\colon \mathbb{R} \to \mathbb{R}$ is continuous, then $f \circ g$ is measurable.

Problem 2: Let $(X, \mathcal{S}, \mu)$ be measure space, and let $f\colon X \to [0, \infty]$ be measurable. Let $a, c \in \mathbb{R}$ with both $a > 0$ and $c > 0$. Show that if $\int f\ d\mu < c$, then $\mu(\{x \in X : f(x) \geq a\}) < \frac{c}{a}$.

Problem 3: Let $f\colon \mathbb{R} \to [0, \infty]$ be a measurable function, and let $t \in \mathbb{R}$. Define $g\colon \mathbb{R} \to [0, \infty]$ by letting $g(x) = f(x + t)$. Show that g is measurable and that

$$\int g\ d\mu = \int f\ d\mu.$$

Problem 4: Let $f\colon \mathbb{R} \to [0, \infty]$ be an integrable function, i.e. assume that f is measurable and that $\int f\ d\mu < \infty$. Define $F\colon \mathbb{R} \to \mathbb{R}$ by letting

$$F(x) = \int_{(-\infty, x]} f\ d\lambda.$$

Show that F is continuous.

Problem 5: Let $(X, \mathcal{S}, \mu)$ be a measure space. Let $E \in \mathcal{S}$ be such that $\mu(E) < \infty$. Let f_n be a sequence of measurable functions such that $f_n \to f$ pointwise on E. Show that for every $\delta > 0$, there exists $A \in \mathcal{S}$ with $A \subseteq E$, $\mu(A) < \delta$, and such that f_n converges to f uniformly on $E \backslash A$.
Hint: We proved a weaker result in class in order to obtain the Bounded Convergence Theorem. Use that weaker result here.
Aside: This result is known as Egorov's Theorem.

Problem 6: A *step function* $g\colon \mathbb{R} \to \mathbb{R}$ is a simple function where the sets are bounded intervals. That is, a step function is a function of the form

$$g = \sum_{j=1}^{m} c_j \cdot \mathbb{I}_{A_j}$$

where the A_j are pairwise disjoint bounded intervals (which may be open, closed, or half-open) and the $c_j \in \mathbb{R}$.
a. Show that for all simple functions $\varphi\colon \mathbb{R} \to \mathbb{R}$ and all $\delta > 0$, there exists a step function $g\colon \mathbb{R} \to \mathbb{R}$ such that $\lambda(\{x \in \mathbb{R} : g(x) \neq \varphi(x)\} < \delta$.
b. Show that for all step functions $g\colon \mathbb{R} \to \mathbb{R}$ and all $\delta > 0$, there exists a continuous function $h\colon \mathbb{R} \to \mathbb{R}$ such that $\lambda(\{x \in \mathbb{R} : h(x) \neq g(x)\} < \delta$.
c. Let $f\colon \mathbb{R} \to \mathbb{R}^*$ be a measurable function. Assume that $\lambda(\{x \in X : f(x) \neq 0\}) < \infty$, that $\mu(\{x \in X : f(x) = \infty\}) = 0$, and that $\mu(\{x \in X : f(x) = -\infty\}) = 0$. Show that for all $\varepsilon > 0$, there exists a continuous function $h\colon \mathbb{R} \to \mathbb{R}$ such that

$$\mu(\{x \in X : |f(x) - h(x)| \geq \varepsilon\}) < \varepsilon.$$