# Homework 7: Due Friday, April 14

**Problem 1:** (from Exercise 3.5.6) Let $(X, \mathcal{S}, \mu)$ be a measure space, and let $T\colon X \to X$ be a measure-preserving transformation. We say that a set $W \subseteq X$ is *wandering* if $T^{-m}(W) \cap T^{-n}(W) = \emptyset$ whenever $m, n \in \mathbb{N}$ with $m \neq n$. Show that $T$ is recurrent if and only if $X$ has no wandering subsets of positive measure.

**Problem 2:** Let $\langle a_n \rangle_{n=1}^{\infty}$ be a bounded sequence of real numbers. Let

$$\ell = \limsup_n a_n = \inf_k \sup_{n \geq k} a_n.$$

Notice that $\ell$ exists because the sequence $\langle \sup_{n \geq k} a_n \rangle_{k=1}^{\infty}$ is decreasing and bounded below.
a. Show that $\langle a_n \rangle$ has a subsequence that converges to $\ell$.
b. Show that if $\langle a_{n_i} \rangle$ is a convergent subsequence of $\langle a_n \rangle$, then $\lim_i a_{n_i} \leq \ell$.
*Aside:* Part a gives another proof of the Bolzano-Weierstrass Theorem.

**Problem 3:** Let $(X, \mathcal{S}, \mu)$ be a measure space, and let $f, g\colon X \to \mathbb{R}^*$ be measurable functions.
a. Show that $\{x \in X : f(x) \neq g(x)\}$ is measurable.
b. Assume that $g(x) \in \mathbb{R}\backslash\{0\}$ for all $x \in X$. Show that $\frac{f}{g}$ is measurable.

**Problem 4:** Let $f\colon [a, b] \to \mathbb{R}$ be a bounded function. Show that if $f$ is Riemann integrable, then $f$ is Lebesgue integrable, and the resulting values (of the corresponding integrals) are equal.

**Problem 5:** Give an example of a function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f$ is not measurable, but $\{x \in \mathbb{R} : f(x) = a\}$ is measurable for each $a \in \mathbb{R}$.

**Problem 6:** Let $(X, \mathcal{S}, \mu)$ be a measure space, and let $f\colon X \to \mathbb{R}^*$ be a measurable function. Let $E = \{x \in X : f(x) \neq 0\}$ be the support of $f$. Assume that $\mu(E) < \infty$, that $\mu(\{x \in X : f(x) = \infty\}) = 0$, and that $\mu(\{x \in X : f(x) = -\infty\}) = 0$.
a. Show that for all $\varepsilon > 0$, there exists $M \in \mathbb{R}$ such that $\mu(\{x \in X : f(x) \notin [-M, M]\}) < \varepsilon$.
b. Show that for all $\varepsilon > 0$ and all $M \in \mathbb{R}$, there exists a simple function $\varphi\colon X \to \mathbb{R}$ such that $|f(x) - \varphi(x)| < \varepsilon$ for all $x \in X$ with $-M \leq f(x) \leq M$.
c. Show that for all $\varepsilon > 0$, there exists a simple function $\varphi\colon X \to \mathbb{R}$ such that

$$\mu(\{x \in X : |f(x) - \varphi(x)| \geq \varepsilon\}) < \varepsilon.$$