Homework 6: Due Friday, March 10

Problem 1: Let (X, d) be a metric space without any isolated points.
a. Show that every nonempty open subset of X is infinite.
b. Let $T\colon X \to X$ be a function. Assume that $x \in X$ has a positive orbit that is dense, i.e. that $\{T^n(x) : n \in \mathbb{N}\}$ is dense in X. Show that for each $k \in \mathbb{N}$, the positive orbit of $T^k(x)$ is dense.
Note: This completes the proof of Theorem 3.3.4.

Problem 2: (from Exercise 3.3.3): Let $T\colon [0,1) \to [0,1)$ be the doubling map. Show that the set of periodic points of T is a dense subset of $[0,1)$.

Problem 3: Show that the doubling map $T\colon [0,1) \to [0,1)$ is topologically transitive by giving a specific example (with justification) of an $x \in [0,1)$ whose positive orbit is dense.

Problem 4: (from Exercise 3.3.1): Let $T\colon [0,1] \to [0,1]$ be the tent map given by

$$T(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq \frac{1}{2} \\ 2 - 2x & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

a. Show that x is a dyadic rational if and only if $T(x)$ is a dyadic rational.
b. If $x \in [0,1]$ is not a dyadic rational, describe the action of T based on the binary representation of x.
c. Show that T is measure-preserving.

Problem 5: (from Exercise 3.3.17): Show that if $T\colon [0,1] \to [0,1]$ is continuous and bijective, then T is not topologically transitive, i.e. there does not exist an $x \in [0,1]$ whose positive orbit is dense.

Problem 6: (from Exercise 3.4.1): Define $T\colon \mathbb{R} \to \mathbb{R}$ by letting

$$T(x) = \begin{cases} x - \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Show that T is measure-preserving.