Homework 5: Due Friday, March 3

Problem 1: (see p. 48) Let μ^* be an outer measure on a nonempty set X. Show that if $A \subseteq X$ is such that $\mu^*(A) = 0$, then

$$\mu^*(C) = \mu^*(A \cap C) + \mu^*(A^c \cap C)$$

for all $C \subseteq X$. In other words, every set with outer measure 0 satisfies the Caratheodory condition for measurability.

Problem 2: (from Exercise 2.5.11 and Exercise 2.5.12) Let $(X, \mathcal{S}, \mu)$ be a measure space. Let $A_1, A_2, A_3, \dots$ be a sequence of measurable sets (i.e. $A_n \in \mathcal{S}$ for all $n \in \mathbb{N}^+$), and let

$$B = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n.$$

a. Show that $B = \{x \in X : \text{There are infinitely many } n \in \mathbb{N}^+ \text{ with } x \in A_n\}$.
b. Show that $B \in \mathcal{S}$.
c. Show that if $\sum_{n=1}^{\infty} \mu(A_n) < \infty$, then $\mu(B) = 0$.
Aside: Part c is known as the Borel-Cantelli Lemma.

Problem 3: (from Exercise 2.5.15) Let $(X, \mathcal{S}, \mu)$ be a measure space. On Homework 4, you showed that the set

$$\mathcal{S}_\mu = \{E \in \mathcal{P}(X) : \text{There exists } A, B \in \mathcal{S} \text{ with } A \subseteq E \subseteq B \text{ and } \mu(B \backslash A) = 0\}$$

was a σ-algebra on X containing $\mathcal{S}$. Define $\overline{\mu} \colon \mathcal{S}_\mu \to [0, \infty]$ as follows. Given $E \in \mathcal{S}_\mu$, fix some $A, B \in \mathcal{S}$ with $A \subseteq E \subseteq B$ and $\mu(B \backslash A) = 0$, and let $\overline{\mu}(E) = \mu(A)$.
a. Show that $\overline{\mu}$ is well-defined.
b. Show that $\overline{\mu}(E) = \mu(E)$ for all $E \in \mathcal{S}$.
c. Show that $(X, \mathcal{S}_\mu, \overline{\mu})$ is a measure space.
d. Show that for all $E \in \mathcal{S}_\mu$ with $\overline{\mu}(E) = 0$, we have $\mathcal{P}(E) \subseteq \mathcal{S}_\mu$.
Note: This shows that $(X, \mathcal{S}_\mu, \overline{\mu})$ is a complete measure space such that $\mathcal{S}_\mu \supseteq \mathcal{S}$ and $\overline{\mu}$ extends μ. The measure space $(X, \mathcal{S}_\mu, \overline{\mu})$ is (shockingly) called the completion of $(X, \mathcal{S}, \mu)$.

Problem 4:
a. Let X and Y be sets, let $\mathcal{S}$ be a σ-algebra on X, let $\mathcal{A} \subseteq \mathcal{P}(Y)$, and let $f \colon X \to Y$. Suppose that $f^{-1}(A) \in \mathcal{S}$ for all $A \in \mathcal{A}$. Show that $f^{-1}(B) \in \mathcal{S}$ for all $B \in \sigma(\mathcal{A})$, where $\sigma(\mathcal{A})$ is the smallest σ-algebra on Y containing $\mathcal{A}$.
b. Given a metric space (X, d), recall that we defined the Borel σ-algebra of X to be the smallest σ-algebra on X that contains every open set. Suppose that (X_1, d_1) and (X_2, d_2) are both metric spaces, and that $f \colon X_1 \to X_2$ is continuous. Show that $f^{-1}(B)$ is a Borel subset of X_1 for all Borel subsets B of X_2.

Problem 5: Let $(X, \mathcal{S}, \mu)$ be a measure space, and let $T \colon X \to X$ be an invertible measurable transformation (see p. 69).
a. Show that $T(A) \in \mathcal{S}$ for all $A \in \mathcal{S}$.
b. Show that T is measure preserving if and only if $\mu(T(A)) = \mu(A)$ for all $A \in \mathcal{S}$.