Homework 4: Due Friday, February 24

Problem 1: (see p. 30) Let $(X_1, \mathcal{S}_1, \mu_1)$ and $(X_2, \mathcal{S}_2, \mu_2)$ be two measure spaces, and assume that $X_1 \cap X_2 = \emptyset$ (otherwise, one can simply rename the elements to make this work). Let $Y = X_1 \cup X_2$ and let

$$\mathcal{T} = \{B \in \mathcal{P}(X_1 \cup X_2) : B \cap X_1 \in \mathcal{S}_1 \text{ and } B \cap X_2 \in \mathcal{S}_2\}.$$

Finally, define $\nu \colon \mathcal{T} \to \mathbb{R}$ by letting $\nu(B) = \mu_1(B \cap X_1) + \mu_2(B \cap X_2)$. Show that $\mathcal{T}$ is a σ-algebra on Y and that $(Y, \mathcal{T}, \nu)$ is a measure space.

Problem 2: (from Exercise 2.6.3) Show that for every null set $A \subseteq \mathbb{R}$, there exists a Borel null set $B \subseteq \mathbb{R}$ with $A \subseteq B$.

Problem 3: (from Exercise 2.6.10 and Exercise 2.6.13) Let $(X, \mathcal{S}, \mu)$ be a measure space. Define a function $d \colon \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ by letting $d(A, B) = \mu(A \triangle B)$. Show that $(\mathcal{S}, d)$ is a pseudometric space.

Problem 4: (from Exercise 2.5.14) Let $(X, \mathcal{S}, \mu)$ be a measure space. Let

$$\mathcal{S}_\mu = \{E \in \mathcal{P}(X) : \text{There exists } A, B \in \mathcal{S} \text{ with } A \subseteq E \subseteq B \text{ and } \mu(B \backslash A) = 0\}.$$

Show that $\mathcal{S}_\mu$ is a σ-algebra containing $\mathcal{S}$.

Problem 5: Let $f \colon \mathbb{R} \to \mathbb{R}$. Show that the set of points of continuity of f, i.e. the set

$$\{a \in \mathbb{R} : f \text{ is continuous at } a\},$$

is a $\mathcal{G}_\delta$ set.
Note: It follows that there is no function that is continuous on $\mathbb{Q}$ and discontinuous on $\mathbb{R} \backslash \mathbb{Q}$.

Definition: Recall the following definitions from class:

- A set $D \subseteq \mathbb{R}$ is *dense* if for all $x, y \in \mathbb{R}$ with $x < y$, we have $D \cap (x, y) \neq \emptyset$.
- A set $A \subseteq \mathbb{R}$ is *nowhere dense* if it is not dense in any interval, i.e. for all $x, y \in \mathbb{R}$ with $x < y$, there exists $c, d \in \mathbb{R}$ with $x < c < d < y$ such that $A \cap (c, d) = \emptyset$.

Problem 6: Let $A \subseteq \mathbb{R}$. Show that the following are equivalent:

1. A is nowhere dense.
2. $\overline{A}$ is nowhere dense.
3. $\overline{A}$ has empty interior, i.e. $\overline{A}$ contains no nontrivial intervals.
4. $\overline{A}^c$ is a dense open set.
5. $A^c = \mathbb{R} \backslash A$ contains a dense open set.