Homework 3: Due Friday, February 17

Problem 1: (from Exercise 2.3.1) Show that if $A \subseteq \mathbb{R}$ is measurable, then for any $t \in \mathbb{R}$, the set $tA = \{ta : a \in A\}$ is measurable.

Problem 2: (from Exercise 2.3.2) Show that if A is a null set, then $A^2 = \{a^2 : a \in A\}$ is also a null set.

Problem 3: (from Exercise 2.3.3) Let $A \subseteq \mathbb{R}$. Show that if there is a measurable set B that differs from A by a null set, i.e. $\lambda^*(A \triangle B) = 0$, then A is measurable.

Problem 4: (from Exercise 2.3.7) Let $A \subseteq \mathbb{R}$. Show that A is measurable if and only if for all $\varepsilon > 0$, there exists a closed set F and an open set G such that $F \subseteq A \subseteq G$ and $\lambda^*(G \backslash F) < \varepsilon$.

Problem 5: (from Exercise 2.4.2) A sequence $A_1, A_2, A_3, \ldots$ of measurable sets is *almost disjoint* if $\lambda(A_i \cap A_j) = 0$ whenever $i \neq j$. Show that if $A_1, A_2, A_3, \ldots$ is an almost disjoint sequence of measurable sets, then $\lambda(\bigcup\limits_{n=1}^{\infty} A_n) = \sum\limits_{n=1}^{\infty} \lambda(A_n)$.

Problem 6: (from Exercise 2.4.3) Let $A \subseteq \mathbb{R}$ be such that $\lambda^*(A) < \infty$. Show that A is measurable if and only if for all $\varepsilon > 0$, there exists a set H that is a finite union of bounded intervals such that $\lambda^*(A \triangle H) < \varepsilon$.

Problem 7: (from Exercise 2.4.4) Show that any collection of pairwise disjoint measurable sets, each of which has positive measure, is countable.