Homework 2: Due Friday, February 10

Problem 1: (from Exercise 2.1.5) Show that if $A, N \subseteq \mathbb{R}$ and $\lambda^*(N) = 0$, then $\lambda^*(A \cup N) = \lambda^*(A)$.

Problem 2: Let $b \in \mathbb{N}$ with $b \geq 2$. Let $A \subseteq \mathbb{R}$ be the set of numbers that have more than one base b representation. Show that $\lambda^*(A) = 0$.

Problem 3: (from Exercise 2.1.4 and Exercise 2.1.9) Let $A \subseteq \mathbb{R}$ and $t \in \mathbb{R}$.
a. Let $A + t = \{a + t : a \in A\}$ be the translation of A by t. Show that $\lambda^*(A + t) = \lambda^*(A)$.
b. Let $tA = \{ta : a \in A\}$. Show that $\lambda^*(tA) = |t| \cdot \lambda^*(A)$.

Problem 4: As mentioned in class, *(Jordan) outer content* c^* is defined in the same way as λ^*, but only using finite covers. In other words

$$c^*(A) = \inf\left\{\sum_{j=1}^{n} |I_j| : A \subseteq \bigcup_{j=1}^{n} I_j \text{ and each } I_j \text{ is a bounded interval}\right\}.$$

Notice that $\lambda^*(A) \leq c^*(A)$ for every set A.
a. Show that $c^*(A) = c^*(\overline{A})$ for every set A, where $\overline{A}$ is the closure of A.
b. Give an example of a set A with $\lambda^*(A) \neq \lambda^*(\overline{A})$
c. Give an example of a set A with $\lambda^*(A) < c^*(A)$.

Problem 5: Let A be the subset of $[0, 1]$ consisting of those numbers that do not contain the digit 3 in their decimal representations. Show that $\lambda^*(A) = 0$.

Problem 6: (from Exercise 2.2.6) Given two sets A and B, let $A + B = \{a + b : a \in A, b \in B\}$. Show that $K + K = [0, 2]$, where K is the Cantor Set.
Hint: Show that $F_n + F_n = [0, 2]$ for all $n \in \mathbb{N}$. It may help to notice that $F_{n+1} = \frac{1}{3}F_n \cup (\frac{2}{3} + \frac{1}{3}F_n)$ for all $n \in \mathbb{N}$.