Homework 10: Due Friday, May 12

Problem 1: (From Exercise 5.2.1) Let $(X, \mathcal{S}, \mu)$ be a probability space, and let $T: X \to X$ be a measure preserving transformation. Suppose that for every $A \in \mathcal{S}$, the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{I}_A(T^k(x))$$

exists and equals $\mu(A)$ a.e. Show that T is ergodic.

Problem 2: Let V be an inner product space.
a. Show that $||\vec{v}+\vec{w}||^2 + ||\vec{v}-\vec{w}||^2 = 2 \cdot ||\vec{v}||^2 + 2 \cdot ||\vec{w}||^2$ for all $\vec{v}, \vec{w} \in V$.
b. Show that $||\vec{v}+\vec{w}||^2 - ||\vec{v}-\vec{w}||^2 = 4 \cdot \langle \vec{v}, \vec{w} \rangle$ for all $\vec{v}, \vec{w} \in V$.
c. Show that the function $f: V \to \mathbb{R}$ defined by $f(\vec{v}) = ||\vec{v}||$ is continuous.
Aside: Part a is known as the *parallelogram law*, because it says that the sums of the squares of the lengths of the sides of a parallelogram equals the sum of the squares of the lengths of the diagonals.

Problem 3: Let V be an inner product space. Let $T: V \to V$ be a linear transformation. Show that the following are equivalent:

1. T preserves the inner product: For all $\vec{u}, \vec{w} \in V$, we have $\langle T(\vec{u}), T(\vec{w}) \rangle = \langle \vec{u}, \vec{w} \rangle$.
2. T preserves the norm: For all $\vec{u} \in V$, we have $||T(\vec{u})|| = ||\vec{u}||$.

A linear transformation with either of (and hence both of) these properties is called an *orthogonal* transformation (or a *unitary* transformation in the complex case).

Problem 4: Let V be an inner product space, and let U be a subspace of V. We saw in class that $U^\perp$ was a subspace of V. Show that $U^\perp$ is a *closed* subspace of V, in the sense that $U^\perp$ is a closed set in the underlying metric space.

Problem 5: Let V be a Hilbert space, and let W be a closed subspace of V. In class, we argued that for each $\vec{v} \in V$, there is a unique $\vec{w} \in W$ minimizing the value $||\vec{v} - \vec{w}||$, i.e. there exists a unique $\vec{w} \in W$ such that $||\vec{v} - \vec{w}|| \leq ||\vec{v} - \vec{x}||$ for all $\vec{x} \in W$. Moreover, for this unique $\vec{w}$, we have $\vec{v} - \vec{w} \in W^\perp$. Define $P: V \to W$ by letting $P(\vec{v})$ be this unique $\vec{w}$. Show that P is a linear transformation.

Problem 6: Let V be a inner product space. Assume that V is separable, i.e. that there is a countable dense set in the underlying metric space. Show that if C is a set of pairwise orthogonal nonzero vectors in V, then C is countable.