Homework 1: Due Friday, February 3

Problem 1: A *dyadic rational* is a element of $\mathbb{Q}$ that can be written in the form $\frac{a}{2^k}$ for some $a \in \mathbb{Z}$ and $k \in \mathbb{N}$. Show that the set of dyadic rationals is dense in $\mathbb{R}$, i.e. that whenever $x, y \in \mathbb{R}$ and $x < y$, there exists a dyadic rational q with $x < q < y$.

Problem 2:
a. Show that $[0, 1]$ can be written as a countable intersection of open sets.
b. Give an example of a countable intersection of open sets that is neither open nor closed.
c. Show that every closed subset of $\mathbb{R}$ can be written as a countable intersection of open sets.

Problem 3: In class, we defined a metric space whose elements were infinite sequences of 0's and 1's. More formally, let X be the set of all functions $f\colon \mathbb{N} \to \{0, 1\}$. Define $d\colon X \times X \to \mathbb{R}$ as follows. Given $f, g\colon \mathbb{N} \to \{0, 1\}$, let $d(f, g) = 0$ if $f(n) = g(n)$ for all $n \in \mathbb{N}$, and otherwise let $d(f, g) = 2^{-m}$, where $m = \min\{n \in \mathbb{N} : f(n) \neq g(n)\}$.
a. Show that $d(f, h) \leq \max\{d(f, g), d(g, h)\}$ for all $f, g, h \in X$.
b. Show that $B(f, \varepsilon)$ is closed for every $f \in X$ and every $\varepsilon > 0$ (we know from Homework 0 that each of these sets is also open).
Note: The triangle inequality condition on metric spaces follows immediately from part a. Metric spaces with this stronger property are called *ultrametric spaces*.

Problem 4: A pseudometric space is a set X together with a function $d\colon X \times X \to \mathbb{R}$ that satisfies the following properties:

- $d(x, x) = 0$ for all $x \in X$ and $d(x, y) \geq 0$ for all $x, y \in X$.
- $d(x, y) = d(y, x)$ for all $x, y \in X$.
- $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Notice that this differs from the definition of a metric space only in that we have the weak inequality $d(x, y) \geq 0$, rather than $d(x, y) > 0$, whenever $x \neq y$.
a. Let (X, d) be a pseudometric space. Define a relation by letting $x \sim y$ mean that $d(x, y) = 0$. Show that $\sim$ is an equivalence relation on X.
b. For each $x \in X$, let $\overline{x} = \{y \in X : x \sim y\}$ be the equivalence class of X. Let $\widetilde{X}$ be the set of equivalence classes of X under $\sim$. Show that the function $\widetilde{d}\colon \widetilde{X} \times \widetilde{X} \to \mathbb{R}$ defined by letting $\widetilde{d}(\overline{x}, \overline{y}) = d(x, y)$ is well-defined.
c. Show that $(\widetilde{X}, \widetilde{d})$ is a metric space.

Problem 5: Let (X_1, d_1) and (X_2, d_2) be metric spaces, and let $f\colon X_1 \to X_2$ be a function. Show that the following are equivalent:

1. For all $x_1 \in X_1$ and all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $y_1 \in X$ with $d_1(x_1, y_1) < \delta$, we have $d_2(f(x_1), f(y_1)) < \varepsilon$.
2. $f^{-1}(D)$ is open in X_1 whenever D is open in X_2.

Note: The first condition is typically how continuity of a function f is defined for metric spaces, so this problem shows that this analytic condition matches the topological one.