Homework 0: Due Friday, January 27

Problem 1: Show that $\log_{10} 2$ is irrational.

Problem 2: Let $a_n \in \mathbb{R}$ with $a_n \geq 0$ for all $n \in \mathbb{N}^+$. Let

$$B = \left\{ \sum_{n=1}^{N} a_n : N \in \mathbb{N}^+ \right\} \qquad \text{and} \qquad C = \left\{ \sum_{n \in F} a_n : F \in \mathcal{P}_{fin}(\mathbb{N}^+) \right\},$$

where $\mathcal{P}_{fin}(\mathbb{N}^+)$ is the set of all finite subsets of $\mathbb{N}^+$.

a. Show that $\sum_{n=1}^{\infty} a_n$ converges if and only if B is bounded above, and that in this case $\sum_{n=1}^{\infty} a_n = \sup B$.

b. Show that B is bounded above if and only if C is bounded above, and that in this case $\sup B = \sup C$.

Note: It follows that $\sum_{n=1}^{\infty} a_n$ converges if and only if C is bounded above, and that in this case $\sum_{n=1}^{\infty} a_n = \sup C$.

Definition: Let $a_n \in \mathbb{R}$ for all $n \in \mathbb{N}^+$. Define two new sequences as follows. For each $n \in \mathbb{N}^+$, let

$$a_n^+ = \begin{cases} a_n & \text{if } a_n \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad a_n^- = \begin{cases} -a_n & \text{if } a_n \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $a_n^+ \geq 0$ and $a_n^- \geq 0$ for all $n \in \mathbb{N}^+$.

Problem 3:

a. Show that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if and only if both $\sum_{n=1}^{\infty} a_n^+$ and $\sum_{n=1}^{\infty} a_n^-$ converge. Moreover, show that in case, we have $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_n^+ - \sum_{n=1}^{\infty} a_n^-$.

b. Show that if $\sum_{n=1}^{\infty} a_n^+ = \infty$ and if $\sum_{n=1}^{\infty} a_n^-$ converges, then $\sum_{n=1}^{\infty} a_n = \infty$.

c. Show that if $\sum_{n=1}^{\infty} a_n^+$ converges and if $\sum_{n=1}^{\infty} a_n^- = \infty$, then $\sum_{n=1}^{\infty} a_n = -\infty$.

Problem 4: Let (X, d) be a metric space (see the definition on page 242). Show that if $x \in X$ and $\varepsilon > 0$, then $B(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ is an open set in X.

Problem 5: Prove Proposition B.1.2 and Proposition B.1.4 on p. 243-244.