Midterm Exam: Due Friday, April 8 at 5:00pm

- You are free to use the posted course notes, the homework solutions, your personal notes, and your previous homework in solving these problems. However, you may *not* consult any other sources (other books, online resources, etc.).
- You may *not* communicate with anybody else (student or otherwise) about the problems on the exam.

Problem 1: (5 points) Let $A = \mathbb{N}$, $B = \{2\}$, and $\mathcal{H} = \{h_1, h_2\}$ where $h_1 \colon A \to A$ is given by $h_1(n) = n^2$ and $h_2 \colon A \to A$ is given by $h_2(n) = n + 4$. Is the simple generating system $(A, B, \mathcal{H})$ free? Explain.

Problem 2: (6 points) Let P be a nonempty set, and let $Form_P^{\#} = G(Sym_P^*, P, \{h_\neg, h_\wedge\})$. Define a function $f \colon Form_P^{\#} \to Form_P^{\#}$ recursively as follows:

- $f(\mathsf{A}) = \mathsf{A}$ for all $\mathsf{A} \in P$.
- $f(\neg\varphi) = f(\varphi) \wedge \neg f(\varphi)$ for all $\varphi \in Form_P^{\#}$.
- $f(\varphi \wedge \psi) = f(\psi) \wedge f(\varphi)$ for all $\varphi, \psi \in Form_P^{\#}$.

Show that $f(\varphi) \vDash \varphi$ for all $\varphi \in Form_P^{\#}$.

Problem 3:. (6 points) Let X and Y be (possibly infinite) sets, and let $R \subseteq X \times Y$. Assume the following:

1. For every $x \in X$, the set $\{y \in Y : (x, y) \in R\}$ is finite.
2. For every finite $X_0 \subseteq X$, there exists an injective $f_0 \colon X_0 \to Y$ such that $(x, f_0(x)) \in R$ for all $x \in X_0$.

Use the Compactness Theorem to show that there exists an injective $f \colon X \to Y$ such that $(x, f(x)) \in R$ for all $x \in X$.
Intuition: For each $x \in X$, the set $\{y \in Y : (x, y) \in R\}$ is the set of all possible partners for x. Thus, this problem asks you to show that if there is a valid pairing for every finite $X_0 \subseteq X$, then there is a valid pairing for all of X.

Problem 4: (10 points) Let $\mathcal{L} = \{\mathsf{R}\}$ where R is a binary relation symbol. Let $\mathcal{M}$ be the $\mathcal{L}$-structure where $M = \mathbb{Z}$ and $\mathsf{R}^{\mathcal{M}} = \{(a, b) \in \mathbb{Z}^2 : a \mid b\}$ (recall that $a \mid b$ means that there exists $m \in \mathbb{Z}$ with $b = am$).
a. (2 points) Show that $\{0\} \subseteq M$ is definable in $\mathcal{M}$.
b. (4 points) Let $P \subseteq \mathbb{N}$ be the set of primes, and let $Q = \{p : p \in P\} \cup \{-p : p \in P\}$. Show that $Q \subseteq M$ is definable in $\mathcal{M}$.
c. (4 points) Show that $\{1\} \subseteq M$ is not definable in $\mathcal{M}$.

Problem 5: (9 points) Let $\mathcal{L} = \{\mathsf{c}, \mathsf{d}, \mathsf{R}\}$ where c and d are constant symbols and R is a binary relation symbol. Let

$$\Sigma = \{\neg(\mathsf{c} = \mathsf{d}), \forall \mathsf{x} \mathsf{R}\mathsf{c}\mathsf{x}, \forall \mathsf{x} \mathsf{R}\mathsf{x}\mathsf{d}, \forall \mathsf{x} \forall \mathsf{y} (\mathsf{R}\mathsf{x}\mathsf{y} \to (\mathsf{x} = \mathsf{c} \vee \mathsf{y} = \mathsf{d}))\}$$

and let $T = Cn(\Sigma)$.
a. (5 points) Show that any two countably infinite models of T are isomorphic.
b. (4 points) Show that T is not complete.

Problem 6: (9 points) Recall the theory RG defined in Definition 6.4.6.
a. (6 points) Show that RG has QE.
b. (3 points) Let $\mathcal{M}$ be a countable model of RG. What are the definable subsets of M? Explain.

Bonus: (2 points) By providing an explicit counterexample, show that if you drop assumption (1) from Problem 3, then the problem is no longer true.