Homework 9: Due Friday, April 29

Problem 1: Let $(W_1, <_1)$ and $(W_2, <_2)$ be well-orderings.
a. Let $W = (W_1 \times \{0\}) \cup (W_2 \times \{1\})$, and define a relation $<$ on W as follows:

1. For any $v, w \in W_1$, we have $(v, 0) < (w, 0)$ if and only if $v <_1 w$.
2. For any $y, z \in W_2$, we have $(y, 1) < (z, 1)$ if and only if $y <_2 z$.
3. For any $w \in W_1$ and $z \in W_2$, we have $(w, 0) < (z, 1)$.

Show that $(W, <)$ is a well-ordering. We call W the *sum* of W_1 and W_2 and denote it by $W_1 \oplus W_2$.
b. Let $W = W_1 \times W_2$, and define a relation $<$ on W as follows. For any $v, w \in W_1$ and $y, z \in W_2$, we have $(v, y) < (w, z)$ if and only if either $v <_1 w$ or ($v = w$ and $y <_2 z$). Show that $(W, <)$ is a well-ordering. We call W the *product* of W_1 and W_2 and denote it by $W_1 \otimes W_2$.

Problem 2: In this problem, α, β, and γ are always ordinals.
a. Show that if $\alpha \leq \beta$, there exists a unique γ such that $\alpha + \gamma = \beta$.
b. Give examples of $\alpha \leq \beta$ such that the equation $\gamma + \alpha = \beta$ has 0, 1, and infinitely many solutions for γ.

Problem 3: Show that if α is an infinite ordinal, then $S(\alpha) \approx \alpha$.

Problem 4: Let $A \subseteq \mathbb{R}$. Suppose that $(A, <)$ is a well-ordering under the usual ordering $<$ on $\mathbb{R}$. Show that A is countable.
Hint: Define an injective function from A to $\mathbb{Q}$.

Problem 5: Show that for every set A, there exists a transitive set T with the following properties:

- $A \subseteq T$.
- $T \subseteq S$ for all transitive sets S with $A \subseteq S$.

T is called the *transitive closure* of A.

Problem 6: Define a transfinite sequence of sets V_α for $\alpha \in \mathbf{ORD}$ by:

1. $V_0 = \emptyset$.
2. $V_{\alpha+1} = \mathcal{P}(V_\alpha)$ for all ordinals α.
3. $V_\alpha = \bigcup\{V_\beta : \beta < \alpha\}$ for all limit ordinals α.

a. Show that V_α is transitive for all ordinals α.
b. Show that if $\beta < \alpha$, then $V_\beta \subseteq V_\alpha$.
c. Show that if $x, y \in V_\omega$, then $\bigcup x$, $\{x, y\}$, and $\mathcal{P}(x)$ are all elements of V_ω.
(Working in ZFC without Foundation, one can show that the Axiom of Foundation is equivalent to the statement that for every set x, there exists an ordinal α with $x \in V_\alpha$.)

Challenge Problems

Problem 1: Show that for every ordinal α, there exists ordinals $\beta_1, \beta_2, \ldots, \beta_k$ with $\alpha \geq \beta_1 > \beta_2 > \cdots > \beta_k$ and $n_1, n_2, \ldots, n_k \in \omega$ such that

$$\alpha = \omega^{\beta_1} \cdot n_1 + \omega^{\beta_2} \cdot n_2 + \cdots + \omega^{\beta_k} \cdot n_k.$$

Furthermore, show that the β_i and n_i in such a representation are unique. This expression of an ordinal in "base ω" is called *Cantor Normal Form.*