Homework 8: Due Friday, April 22

In each of these problems, use only results from the notes together with justification from the axioms. You do not need to be perfectly precise, but (for example) do not use any “obvious” facts about the natural numbers that we have not justified.

Problem 1: Let A and B be sets. Suppose that $a \in A$ and $b \in B$. Show that $(a, b) \in \mathcal{P}(\mathcal{P}(A \cup B))$ where $\mathcal{P}(x)$ is the power set of x. This allows one to construct $A \times B$ using Power Set (and Separation) instead of Collection.

Problem 2: Show that the class of all ordered pairs is a proper class.

Problem 3:
a. Let $m, n \in \omega$. Prove that if $S(m) = S(n)$, then $m = n$.
b. Show that for every $n \in \omega$ with $n \neq 0$, there exists a unique $m \in \omega$ with $S(m) = n$.
c. Prove that $m + n = n + m$ for all $m, n \in \omega$.
Hint for c: It may help to first prove that $1 + n = S(n)$ for every $n \in \omega$.

Problem 4: Induction Variants.
a. Let $X \subseteq \omega$ and let $k \in \omega$. Suppose that $k \in X$ and whenever $n \in X$, we have $S(n) \in X$. Show that $n \in X$ for all $n \in \omega$ with $n \geq k$.
b. Let $X \subseteq \omega$ and let $k \in \omega$. Suppose that $0 \in X$ and whenever $n \in X$ and $n < k$, we have $S(n) \in X$. Show that for all $n \in \omega$, if $n \leq k$, then $n \in X$.
c. Let $X \subseteq \omega \times \omega$. Suppose that for all $(m, n) \in \omega \times \omega$, if $(k, \ell) \in X$ for all $k, \ell \in \omega$ with either $k < m$ or ($k = m$ and $\ell < n$), then $(m, n) \in X$. Prove that $X = \omega \times \omega$.

Problem 5:
a. Show that if A is a finite set and $b \notin A$, then $A \cup \{b\}$ is finite with $|A \cup \{b\}| = |A| + 1$.
b. Show that if A and B are finite and disjoint (i.e. $A \cap B = \emptyset$), then $A \cup B$ is finite with $|A \cup B| = |A| + |B|$.

Problem 6:
a. Explain how to define the function $f(n) = 2^n$ formally using Theorem 8.5.1.
b. Show that if A is finite with $|A| = n$, then $\mathcal{P}(A)$ is finite with $|\mathcal{P}(A)| = 2^n$.