Homework 7: Due Friday, April 15

Required Problems

Problem 1: Let $\mathcal{L} = \{\mathsf{R}\}$ where R is a binary relation symbol.
a. Show that the class of directed acyclic graphs is not an elementary class in the language $\mathcal{L}$. (You showed that it was a weak elementary class in Homework 4).
b. Show that the class of all connected graphs is not a weak elementary class in the language $\mathcal{L}$. (A graph (V, E) is *connected* if whenever $v, w \in V$ are distinct, there exists $u_1, u_2, \dots, u_n \in V$ such that $u_1 = v$, $u_n = w$, and $(u_i, u_{i+1}) \in E$ for all i with $1 \le i \le n - 1$).

Problem 2: Let $\mathcal{L} = \{\mathsf{e}, \mathsf{f}\}$ where e is a constant symbol and f is a binary function symbol.
a. Show that the class of all simple abelian groups is not a weak elementary class in the language $\mathcal{L}$. (Recall that an abelian group is simple if and only if it has no proper nontrivial subgroup, since all subgroups are normal.)
b. Show that the class of all torsion-free abelian groups (that is, abelian groups in which every nonidentity element has infinite order) is a weak elementary class but not an elementary class in the language $\mathcal{L}$.

Problem 3: Let $\mathcal{L} = \{\mathsf{R}\}$ where R is a binary relation symbol. Let $\mathcal{Q}$ be the $\mathcal{L}$-structure $(\mathbb{Q}, <)$. Suppose that $\mathcal{M}$ is an infinite $\mathcal{L}$-structure which is a model of

$$\Sigma = \{\forall \mathsf{x} \neg \mathsf{Rxx}, \forall \mathsf{x} \forall \mathsf{y} \forall \mathsf{z}((\mathsf{Rxy} \wedge \mathsf{Ryz}) \to \mathsf{Rxz}), \forall \mathsf{x} \forall \mathsf{y}(\mathsf{Rxy} \vee \mathsf{Ryx} \vee \mathsf{x} = \mathsf{y})\}$$

(i.e. $\mathcal{M}$ is an infinite strict linear ordering). Show that there exists a model $\mathcal{N}$ of Σ such that $\mathcal{M} \equiv \mathcal{N}$ and such that there exists an embedding from $\mathcal{Q}$ to $\mathcal{N}$.

Stated more succinctly, show that for every infinite linear ordering, there exists an elementarily equivalent linear ordering which embeds the rationals.

Problem 4: Let $\mathcal{L} = \{\mathsf{0}, \mathsf{1}, +, \cdot, <\}$ be the language of arithmetic. For each $n \in \mathbb{N}$, let $\varphi_n(\mathsf{x})$ be the formula $\exists \mathsf{y}(\underline{n} \cdot \mathsf{y} = \mathsf{x})$.
a. Show that for any nonstandard model of arithmetic $\mathcal{M}$, there exists $a \in M \backslash \{\mathsf{0}^{\mathcal{M}}\}$ such that $(\mathcal{M}, a) \vDash \varphi_n$ for all $n \in \mathbb{N}^+$.
b. Let $P \subseteq \mathbb{N}$ be the set of primes numbers and suppose that $Q \subseteq P$. Show that there exists a countable nonstandard model of arithmetic $\mathcal{M}$ and an $a \in M$ with both of the following properties:

- $(\mathcal{M}, a) \vDash \varphi_p$ for all $p \in Q$.
- $(\mathcal{M}, a) \vDash \neg\varphi_p$ for all $p \in P \backslash Q$.

Challenge Problems

Problem 1: Let $\mathcal{M}$ be a nonstandard model of arithmetic. Show that $\{\underline{n}^{\mathcal{M}} : n \in \mathbb{N}\} \subseteq M$ is not definable in $\mathcal{M}$.