Homework 6: Due Friday, March 18

Required Problems

Problem 1: Using the fact that DLO has QE, determine (with proof) all definable subsets of $\mathbb{Q}^2$ in the structure $(\mathbb{Q}, <)$.

Problem 2: Let $\mathcal{L} = \{\mathsf{R}\}$ where R is a binary relation symbol. Consider the $\mathcal{L}$-structure $\mathcal{M}$ that is the linear ordering obtained by putting one copy of $\mathbb{R}$ after another. More formally, $M = (\mathbb{R} \times \{0\}) \cup (\mathbb{R} \times \{1\})$, where we order elements as usual in each copy, and where $(a, 0) < (b, 1)$ for all $a, b \in \mathbb{R}$. Show that $\mathcal{M}$ is not isomorphic to $(\mathbb{R}, <)$.
Note: If we do the same construction with $\mathbb{Q}$, then the resulting linear ordering is a countable model of DLO, so is isomorphic to $(\mathbb{Q}, <)$.

Problem 3: Let $\mathcal{L} = \{\mathsf{0}, \mathsf{1}, +\}$ where $\mathsf{0}$ and $\mathsf{1}$ are constant symbols, and $+$ is a binary function symbol. Let $\mathcal{M}$ be the $\mathcal{L}$-structure where $M = \mathbb{Z}$, and where the symbols are interpreted in the usual way. Let $T = Th(\mathcal{M})$.
a. Show that if $X \subseteq \mathbb{Z}$ is definable in $\mathcal{M}$ by a quantifier-free formula, then X is either finite or cofinite.
b. Show that T does not have QE by finding a definable subset of $\mathbb{Z}$ that is neither finite nor cofinite.

Problem 4: Let $\mathcal{L} = \{\mathsf{R}\}$ where R is a binary relation symbol. For each $n \in \mathbb{N}^+$, let σ_n be the sentence

$$\forall \mathsf{y} \exists \mathsf{x}_1 \exists \mathsf{x}_2 \cdots \exists \mathsf{x}_\mathsf{n} \Big(\bigwedge_{1 \le \mathsf{i} < \mathsf{j} \le \mathsf{n}} \neg(\mathsf{x}_\mathsf{i} = \mathsf{x}_\mathsf{j}) \wedge \bigwedge_{\mathsf{i}=1}^{\mathsf{n}} \mathsf{R}\mathsf{x}_\mathsf{i}\mathsf{y} \Big)$$

and let τ_n be the sentence

$$\exists \mathsf{x}_1 \exists \mathsf{x}_2 \cdots \exists \mathsf{x}_\mathsf{n} \Big(\bigwedge_{1 \le \mathsf{i} < \mathsf{j} \le \mathsf{n}} \neg(\mathsf{x}_\mathsf{i} = \mathsf{x}_\mathsf{j}) \wedge \bigwedge_{1 \le \mathsf{i} < \mathsf{j} \le \mathsf{n}} \neg \mathsf{R}\mathsf{x}_\mathsf{i}\mathsf{x}_\mathsf{j} \Big)$$

Finally, let

$$\Sigma = \{\forall \mathsf{x} \mathsf{R}\mathsf{x}\mathsf{x}, \forall \mathsf{x} \forall \mathsf{y} (\mathsf{R}\mathsf{x}\mathsf{y} \to \mathsf{R}\mathsf{y}\mathsf{x}), \forall \mathsf{x} \forall \mathsf{y} \forall \mathsf{z} ((\mathsf{R}\mathsf{x}\mathsf{y} \wedge \mathsf{R}\mathsf{y}\mathsf{z}) \to \mathsf{R}\mathsf{x}\mathsf{z})\} \cup \{\sigma_n : n \in \mathbb{N}^+\} \cup \{\tau_n : n \in \mathbb{N}^+\}$$

and let $T = Cn(\Sigma)$. Notice that models of T are equivalence relations such that there are infinitely many equivalence classes, and such that every equivalence class is infinite.
a. Show that any two countable models of T are isomorphic, and hence that T is complete.
b. Let $\mathcal{M}$ be the $\mathcal{L}$-structure where $M = \{n \in \mathbb{N} : n \geq 2\}$ and $\mathsf{R}^{\mathcal{M}} = \{(a, b) \in M^2 :$ For all primes p, we have $p \mid a$ if and only if $p \mid b\}$. Let $\mathcal{N}$ be the $\mathcal{L}$-structure where $N = \mathbb{R}^2$ and $\mathsf{R}^{\mathcal{N}} = \{((a_1, b_1), (a_2, b_2)) \in N^2 : a_2 - a_1 = b_2 - b_1\}$. Show that $\mathcal{M} \equiv \mathcal{N}$ and $\mathcal{M} \not\cong \mathcal{N}$.
c. Show that T has QE.

Problem 5: Let $\mathcal{L} = \{\mathsf{f}\}$, where f is a unary function symbol. For each $n \in \mathbb{N}^+$, let σ_n be the sentence $\forall \mathsf{x} \neg (\mathsf{f}\mathsf{f} \cdots \mathsf{f}\mathsf{x} = \mathsf{x})$, where there are n many f's. Let

$$\Sigma = \{\forall \mathsf{x} \forall \mathsf{y} (\mathsf{f}\mathsf{x} = \mathsf{f}\mathsf{y} \to \mathsf{x} = \mathsf{y}), \forall \mathsf{y} \exists \mathsf{x} (\mathsf{f}\mathsf{x} = \mathsf{y})\} \cup \{\sigma_n : n \in \mathbb{N}^+\}.$$

Let $T = Cn(\Sigma)$. Thus, models of T are structures where f is interpreted as a bijection without any finite cycles. For example, the structure $\mathcal{M}$ with universe $M = \mathbb{Z}$ and with $\mathsf{f}^{\mathcal{M}}(a) = a + 1$ for all $n \in \mathbb{Z}$ is a model of T.

a. Show that all models of T are infinite.
b. Give an example of a model of T that is not isomorphic to the example described above.
c. Show that T has QE.

Problem 6: Let $\mathcal{L} = \{\mathsf{R}\}$ where R is a binary relation symbol, and let $\mathsf{x}, \mathsf{y} \in Var$ be distinct.
a. Give a deduction showing that $\exists \mathsf{x} \forall \mathsf{y} \mathsf{R} \mathsf{x} \mathsf{y} \vdash \forall \mathsf{y} \exists \mathsf{x} \mathsf{R} \mathsf{x} \mathsf{y}$.
b. Show that $\forall \mathsf{y} \exists \mathsf{x} \mathsf{R} \mathsf{x} \mathsf{y} \nvdash \exists \mathsf{x} \forall \mathsf{y} \mathsf{R} \mathsf{x} \mathsf{y}$.

Problem 7: Let $\mathcal{L}$ be the basic group language, and let Σ be the group theory axioms, i.e.

$$\Sigma = \{\forall \mathsf{x} \forall \mathsf{y} \forall \mathsf{z}(\mathsf{fxfyz} = \mathsf{ffxyz}), \forall \mathsf{x}(\mathsf{fex} = \mathsf{x} \wedge \mathsf{fxe} = \mathsf{x}), \forall \mathsf{x} \exists \mathsf{y}(\mathsf{fxy} = \mathsf{e} \wedge \mathsf{fyx} = \mathsf{e})\}.$$

Give a deduction showing that $\Sigma \vdash \forall \mathsf{x}(\mathsf{fex} = \mathsf{e} \rightarrow \mathsf{x} = \mathsf{e})$.