# Homework 5: Due Friday, March 4

## Required Problems

**Problem 1:** Let $\mathcal{L} = \{\mathsf{P}\}$ where $\mathsf{P}$ is a unary relation symbol. Let $\mathcal{M}$ be the $\mathcal{L}$-structure where $M = \mathbb{Q}$ and $\mathsf{P}^{\mathcal{M}} = \{q \in \mathbb{Q} : q > 0\}$. Let $\mathcal{N}$ be the $\mathcal{L}$-structure where $N = \mathbb{N}$ and $\mathsf{P}^{\mathcal{N}} = \{2n : n \in \mathbb{N}\}$. Show that $\mathcal{M} \cong \mathcal{N}$.

**Problem 2:**
a. Let $\mathcal{L} = \{\mathsf{P}\}$ where $\mathsf{P}$ is a unary relation symbol. Calculate $I(Cn(\emptyset), n)$ for all $n \in \mathbb{N}^+$.
b. Let $\mathcal{L} = \{\mathsf{R}\}$ where $\mathsf{R}$ is a binary relation symbol, and let $\sigma$ be the sentence

$$(\forall \mathsf{x} \forall \mathsf{y}(\mathsf{Rxy} \rightarrow \mathsf{Ryx})) \wedge \forall \mathsf{x}((\exists \mathsf{y} \mathsf{Rxy}) \rightarrow (\forall \mathsf{y} \mathsf{Rxy})).$$

Calculate $I(Cn(\sigma), n)$ for all $n \in \mathbb{N}^+$.

**Problem 3:**
a. Let $\mathcal{L} = \{\mathsf{P}\}$ where $\mathsf{P}$ is a unary relation symbol and let $\mathcal{M}$ be a finite $\mathcal{L}$-structure. Show that there exists $\sigma \in Sent_{\mathcal{L}}$ such that for all $\mathcal{L}$-structures $\mathcal{N}$, we have

$$\mathcal{N} \vDash \sigma \text{ if and only if } \mathcal{M} \cong \mathcal{N}.$$

b. Solve part (a) in the case of $\mathcal{L} = \{\mathsf{R}\}$, where $\mathsf{R}$ is a binary relation symbol .
*Note:* This problem generalizes to any finite language $\mathcal{L}$. In particular, if $\mathcal{L}$ is a finite language and $\mathcal{M}$ is a finite $\mathcal{L}$-structure, then for any $\mathcal{L}$-structure $\mathcal{N}$, we have $\mathcal{M} \equiv \mathcal{N}$ if and only if $\mathcal{M} \cong \mathcal{N}$. In other words, if $\mathcal{M}$ is finite, then being elementarily equivalent to $\mathcal{M}$ is the same thing as being isomorphic to $\mathcal{M}$. In contrast, we will show that this is *never* the case for an infinite $\mathcal{L}$-structure $\mathcal{M}$.

**Definition:** A set $A \subseteq \mathbb{N}^+$ is called a *spectrum* if there exists a finite language $\mathcal{L}$ and $\sigma \in Sent_{\mathcal{L}}$ such that $A = Spec(\sigma)$.

**Problem 4:**
a. Show that every finite set $F \subseteq \mathbb{N}^+$ is a spectrum.
b. Show that $\{2n + 1 : n \in \mathbb{N}\}$ is a spectrum.
c. Show that $\{n \in \mathbb{N}^+ : n > 1 \text{ and } n \text{ is composite}\}$ is a spectrum.

**Problem 5:**
a. Show that if $\mathcal{L}$ is a finite language and $\sigma, \tau \in Sent_{\mathcal{L}}$, then $Spec(\sigma) \cup Spec(\tau) = Spec(\sigma \vee \tau)$.
b. Show that there exists a finite language $\mathcal{L}$ and $\sigma, \tau \in Sent_{\mathcal{L}}$ such that $Spec(\sigma) \cap Spec(\tau) \neq Spec(\sigma \wedge \tau)$.
c. Show that there exists a finite language $\mathcal{L}$ and $\sigma \in Sent_{\mathcal{L}}$ such that $\mathbb{N}^+ \backslash Spec(\sigma) \neq Spec(\neg\sigma)$.
d. Show that if $A, B \subseteq \mathbb{N}^+$ are both spectra, then $A \cap B$ is a spectrum.
*Cultural Aside:* Although the class of spectra is closed under finite unions and intersections (by parts (a) and (d)), it is an open question whether the class of spectra is closed under complement. This problem is closely tied to problems in complexity theory. As mentioned in class, it turns out that the class of spectra is exactly the collection of subsets of $\mathbb{N}^+$ which are in the complexity class $NE$, i.e. those accepted by a nondeterministic Turing machine which runs in time $2^{O(n)}$. Thus, the question of whether the class of spectra is closed under complement is equivalent to whether NE = co-NE, the analogue of the question of whether NP = co-NP at a slightly higher complexity level.

## Challenge Problems

**Problem 1:**
a. Show that $\{n^2 : n \in \mathbb{N}^+\}$ is a spectrum.
b. Show that $\{p \in \mathbb{N}^+ : p \text{ is prime}\}$ is a spectrum.