Homework 4: Due Friday, February 26

Required Problems

Problem 1:
a. Let $\mathcal{L} = \{\mathsf{f}\}$ where f is a unary function symbol. Show that the class of all $\mathcal{L}$-structures $\mathcal{M}$ such that $\mathsf{f}^{\mathcal{M}}$ is a bijection on M is an elementary class in the language $\mathcal{L}$.
b. A *directed graph* is a nonempty set V of vertices together with a set $E \subseteq V \times V$ where $(u, w) \in E$ intuitively represents an edge originating at u and terminating at w. A cycle in a directed graph is a sequence $u_1 u_2 \cdots u_k$ of vertices, such that $(u_i, u_{i+1}) \in E$ for $1 \leq i \leq k-1$ and $(u_k, u_1) \in E$. If we let $\mathcal{L} = \{\mathsf{R}\}$, where R is a binary relation symbol, then directed graphs correspond exactly to $\mathcal{L}$-structures. Show that the class of directed acyclic graphs (that is, directed graphs with no cycles) is a weak elementary class in this language.

Problem 2:
a. Let $\mathcal{L} = \{\mathsf{f}\}$ where f is a binary function symbol. Show that $(\mathbb{N}, +) \not\equiv (\mathbb{Z}, +)$.
b. Let $\mathcal{L} = \{\mathsf{f}\}$ where f is a binary function symbol. Define $g \colon \{1, 2, 3, 4\}^2 \to \{1, 2, 3, 4\}$ and $h \colon \{a, b, c, d\}^2 \to \{a, b, c, d\}$ by

g	1	2	3	4
1	4	3	1	1
2	2	2	1	2
3	1	4	1	4
4	1	3	2	3

h	a	b	c	d
a	b	b	c	b
b	a	d	d	a
c	b	a	c	a
d	d	b	c	a

Interpret the diagrams as follows. If $m, n \in \{1, 2, 3, 4\}$, to calculate the value of $g(m, n)$, go to row m and column n. For example, $g(1, 2) = 3$. Similarly for h. Show that $(\{1, 2, 3, 4\}, g) \not\equiv (\{a, b, c, d\}, h)$.
c. Let $\mathcal{L} = \{\mathsf{R}\}$ where R is a 3-ary relation symbol. Let $\mathcal{M}$ be the $\mathcal{L}$-stucture where $M = \mathbb{R}$ and $\mathsf{R}^{\mathcal{M}}$ is the "betweeness relation", i.e. $\mathsf{R}^{\mathcal{M}} = \{(a, b, c) \in \mathbb{R}^3 : \text{Either } a \leq b \leq c \text{ or } c \leq b \leq a\}$. Let $\mathcal{N}$ be the $\mathcal{L}$-stucture where $N = \mathbb{R}^2$ and $\mathsf{R}^{\mathcal{N}}$ is the "collinearity relation", i.e. $\mathsf{R}^{\mathcal{N}} = \{((x_1, y_1), (x_2, y_2), (x_3, y_3)) \in (\mathbb{R}^2)^3 : \text{There exists } a, b, c \in \mathbb{R} \text{ with either } a \neq 0 \text{ or } b \neq 0 \text{ such that } ax_i + by_i = c \text{ for all } i\}$. Show that $\mathcal{M} \not\equiv \mathcal{N}$.

Problem 3: Let $\mathcal{L} = \{\mathsf{f}\}$ where f is a binary function symbol. Let $\mathcal{M}$ be the $\mathcal{L}$-structure where $M = \{0, 1\}^*$ and $\mathsf{f}^{\mathcal{M}} \colon M^2 \to M$ is concatenation (i.e. $\mathsf{f}^{\mathcal{M}}(\sigma, \tau) = \sigma\tau$).
a. Show that $\{\lambda\} \subseteq M$ is definable in $\mathcal{M}$.
b. Show that for each $n \in \mathbb{N}$, the set $\{\sigma \in M : |\sigma| = n\}$ is definable in $\mathcal{M}$.
c. Find all automorphisms of $\mathcal{M}$.
d. Show that $\{\sigma \in M : \sigma \text{ contains no 1's}\} = \{0\}^*$ is not definable in $\mathcal{M}$.

Problem 4: Let $\mathcal{L} = \{\mathsf{f}\}$ where f is a binary function symbol. Let $\mathcal{M}$ be the $\mathcal{L}$-structure where $M = \mathbb{N}$ and $\mathsf{f}^{\mathcal{M}} \colon M^2 \to M$ is multiplication (i.e. $\mathsf{f}^{\mathcal{M}}(m, n) = m \cdot n$).
a. Show that $\{0\} \subseteq M$ is definable in $\mathcal{M}$.
b. Show that $\{1\} \subseteq M$ is definable in $\mathcal{M}$.
c. Show that $\{p \in M : p \text{ is prime}\}$ is definable in $\mathcal{M}$.
d. Find all automorphisms of $\mathcal{M}$.
e. Show that $\{n\} \subseteq M$ is not definable in $\mathcal{M}$ whenever $n \geq 2$.
f. Show that $\{(k, m, n) \in M^3 : k + m = n\}$ is not definable in $\mathcal{M}$.

Problem 5: Let $\mathcal{L} = \{\mathsf{e}, \mathsf{f}\}$ be the basic group theory language. Let $\mathcal{M}$ be the symmetric group S_4, and let $X \subseteq M$ be the set of all transpositions (i.e. 2-cycles) in S_4. Show that X is definable in $\mathcal{M}$. If you give an explicit formula, you should explain why it works.