Homework 3: Due Friday, February 19

Required Problems

Problem 1:
a. For each $\varphi, \psi \in Form_P$, give a deduction showing that $\neg\varphi \vdash \neg(\varphi \wedge \psi)$.
b. For each $\varphi, \psi \in Form_P$, give a deduction showing that $\neg(\varphi \wedge \psi) \vdash (\neg\varphi) \vee (\neg\psi)$.

Problem 2:
a. In the proof of the Soundness Theorem (Theorem 3.5.2), we showed how statement (2) followed easily from statement (1). Show that this implication can be reversed. That is, give a short proof that "If $\Gamma \vdash \varphi$, then $\Gamma \vDash \varphi$" using the assumption "Every satisfiable set of formulas is consistent".
b. In the proof of the Completeness Theorem (Theorem 3.5.2), we showed how statement (2) followed easily from statement (1). Show that this implication can be reversed. That is, give a short proof that "Every consistent set of formulas is satisfiable" using the assumption "If $\Gamma \vDash \varphi$, then $\Gamma \vdash \varphi$".
c. Show how to obtain each version of the Compactness Theorem from the other using only short semantic arguments.

Problem 3: Suppose that $\theta \vdash \gamma$ and $\gamma \vdash \theta$. Show that if $\Gamma \vdash \varphi$, then $\Gamma \vdash Subst_\gamma^\theta(\varphi)$.
Hint: You don't need to stay on the syntactic side. Don't be a hero.

Problem 4:
a. Suppose that we eliminate the $\to I$ rule, the $\neg PC$ rule, and the $Contr$ rule. Show that the Completeness Theorem no longer holds.
b. Suppose that we eliminate the $\neg PC$ rule and the $Contr$ rule. Show that the Completeness Theorem no longer holds.
Hint: For part a, notice that these 3 rules are the only ones that let us completely remove a formula from the left-hand side. State precisely what this gives you, and how it establishes the result. Part (b) will require a bit more insight, but use the idea of part (a) as a guide.

Problem 5: Given two partial orderings $\leq$ and $\leq'$ on a set B, we say that $\leq'$ *extends* $\leq$ if whenever $a \leq b$, we have $a \leq' b$. In other words, the set of ordered paris that satisfies the relation $\leq$ is a subset of the set of ordered pairs that satisfies $\leq'$. Recall that a linear ordering is a partial ordering with the extra properties that for all $a, b \in B$, either $a \leq b$ or $b \leq a$.
a. Show that if $(B, \leq)$ is a *finite* partial ordering (i.e. where B is finite), then there exists a linear ordering $\leq'$ of B extending $\leq$.
b. Show that if $(B, \leq)$ is any partial ordering, then there exists a linear ordering $\leq'$ of B extending $\leq$.

Challenge Problems

There are 2 challenge problems, each of which asks you to give a proof of the Compactness Theorem without using the notion of syntactic implication. The first mimics the essential ideas in the proof of Completeness given in class, but stays semantic the whole way. The second uses ideas and results from topology.

Problem 1: We say a set $\Gamma \subseteq Form_P$ is *finitely satisfiable* if every finite subset of Γ is satisfiable.
a. Show that if $\Gamma \subseteq Form_P$ is finitely satisfiable and $\varphi \in Form_P$, then either $\Gamma \cup \{\varphi\}$ is finitely satisfiable or $\Gamma \cup \{\neg\varphi\}$ is finitely satisfiable.

b. Show that if $\Gamma \subseteq Form_P$ is finitely satisfiable, then there exists $\Delta \supseteq \Gamma$ which is both complete and finitely satisfiable.
c. Show that if Δ is both complete and finitely satisfiable, then Δ is satisfiable.
d. Establish the Compactness Theorem.

Problem 2: For any set P, let T_P be the topological space $\{0,1\}^P$ (which can be viewed either as the product of copies of $\{0,1\}$ indexed by P, or as truth assignments on P) where we give $\{0,1\}$ the discrete topology and $\{0,1\}^P$ the corresponding product topology.
a. Viewing elements of T_P as truth assignments, show that the set $\{M \in T_P : v_M(\varphi) = 1\}$ is closed for all $\varphi \in Form_P$.
b. Use Tychonov's Theorem to prove the Compactness Theorem.
c. Without applying Tychonov's Theorem, use the Compactness Theorem to prove that T_P is compact for every P.