Homework 2: Due Friday, February 12

Required Problems

Problem 1: Definition 3.1.16 defines a function $Subform\colon Form_P \to \mathcal{P}(Form_P)$ recursively as follows:

- $Subform(\mathsf{A}) = \{\mathsf{A}\}$ for all $\mathsf{A} \in P$.
- $Subform(\neg\varphi) = \{\neg\varphi\} \cup Subform(\varphi)$.
- $Subform(\Diamond\varphi\psi) = \{\Diamond\varphi\psi\} \cup Subform(\varphi) \cup Subform(\psi)$ for each $\Diamond \in \{\wedge, \vee, \to\}$.

Suppose that $\varphi, \psi \in Form_P$ and that ψ is a substring of φ (i.e. there exists $\theta, \rho \in Sym_P^*$ such that $\varphi = \theta\psi\rho$). Show that $\psi \in Subform(\varphi)$.

Problem 2: Given any $\theta, \gamma \in Form_P$, Definition 3.1.17 describes a function $Subst_\gamma^\theta\colon Form_P \to Form_P$ (intuitively substituting θ for all occurrences of γ), which is defined recursively as follows:

- $Subst_\gamma^\theta(\mathsf{A}) = \begin{cases} \theta & \text{if } \gamma = \mathsf{A} \\ \mathsf{A} & \text{otherwise.} \end{cases}$
- $Subst_\gamma^\theta(\neg\varphi) = \begin{cases} \theta & \text{if } \gamma = \neg\varphi \\ \neg Subst_\gamma^\theta(\varphi) & \text{otherwise.} \end{cases}$
- $Subst_\gamma^\theta(\Diamond\varphi\psi) = \begin{cases} \theta & \text{if } \gamma = \Diamond\varphi\psi \\ \Diamond Subst_\gamma^\theta(\varphi) Subst_\gamma^\theta(\psi) & \text{otherwise.} \end{cases}$

  for each $\Diamond \in \{\wedge, \vee, \to\}$.

Show that if $M\colon P \to \{0,1\}$ is a truth assignment with $v_M(\theta) = v_M(\gamma)$, then $v_M(\varphi) = v_M(Subst_\gamma^\theta(\varphi))$ for every $\varphi \in Form_P$.

Problem 3: Let $Form_P^- = G(Sym_P^*, P, \{h_\neg, h_\wedge, h_\vee\})$. Define a function $Dual\colon Form_P^- \to Form_P^-$ recursively as follows:

- $Dual(\mathsf{A}) = \neg\mathsf{A}$ for all $\mathsf{A} \in P$.
- $Dual(\neg\varphi) = \neg Dual(\varphi)$.
- $Dual(\wedge\varphi\psi) = \vee Dual(\varphi) Dual(\psi)$.
- $Dual(\vee\varphi\psi) = \wedge Dual(\varphi) Dual(\psi)$.

Show that $Dual(\varphi)$ is semantically equivalent to $\neg\varphi$ for all $\varphi \in Form_P^-$.

We can extend the notion of semantic equivalence to sets of formulas.

Definition: Let $\Gamma_1, \Gamma_2 \subseteq Form_P$. We say that Γ_1 and Γ_2 are *semantically equivalent* if $\Gamma_1 \vDash \gamma_2$ for all $\gamma_2 \in \Gamma_2$ and $\Gamma_2 \vDash \gamma_1$ for all $\gamma_1 \in \Gamma_1$. Notice that this is equivalent to saying that whenever $M\colon P \to \{0,1\}$ is a truth assignment, then $v_M(\gamma_1) = 1$ for all $\gamma_1 \in \Gamma_1$ if and only if $v_M(\gamma_2) = 1$ for all $\gamma_2 \in \Gamma_2$.

Definition: Let $\Gamma \subseteq Form_P$. We say that Γ is *independent* if there is no $\varphi \in \Gamma$ such that $\Gamma\backslash\{\varphi\} \vDash \varphi$. Notice that this is equivalent to saying that Γ is not semantically equivalent to any proper subset.

Problem 4:
a. Show that if Γ is finite, then Γ has an independent semantically equivalent subset.
b. Show that there exists a set P and an infinite set $\Gamma \subseteq Form_P$ which has no independent semantically equivalent subset.

Problem 5: Let $P = \{\mathsf{A}_0, \mathsf{A}_1, \mathsf{A}_2, \mathsf{A}_3, \mathsf{A}_4, \mathsf{A}_5, \mathsf{A}_6\}$. Show that there exists a boolean function $f \colon \{0,1\}^7 \to \{0,1\}$ such that $Depth(\varphi) \geq 5$ for all φ with $B_\varphi = f$.
Hint: Do a counting argument.

Problem 6: For each $\varphi \in Form_P$, give a deduction showing that $\varphi \vdash \neg\neg\varphi$.

Challenge Problems

Problem 1: Show that every countable set is semantically equivalent to an independent set.