Homework 11: Due Friday, May 13

Required Problems

Problem 1: Let X be a set, and let $\mathcal{U}$ be an ultrafilter on X. Show that if $\mathcal{T} \subseteq \mathcal{P}(X)$ is finite, and $\bigcup \mathcal{T} \in \mathcal{U}$, then there exists $A \in \mathcal{T}$ such that $A \in \mathcal{U}$. Furthermore, show that if the elements of $\mathcal{T}$ are pairwise disjoint, then the A is unique.

Problem 2: Show that every ultrafilter on a finite set is principal.

Problem 3: We say that an ultrafilter $\mathcal{U}$ on a set X is σ-complete if $\bigcap_{n\in\omega} A_n \in \mathcal{U}$ whenever $A_n \in \mathcal{U}$ for all $n \in \omega$. Show that an ultrafilter on ω is σ-complete if and only if it is principal. (The question of whether there exists a nonprincipal σ-complete ultrafilter on any set is a very deep and interesting one. If such an ultrafilter exists on a set X, then X is ridiculously large.)

Problem 4: Let $L = \{\mathsf{P}\}$ where P is a unary relation symbol. For each $n \in \mathbb{N}^+$, let

$$\sigma_n = \exists \mathsf{x}_1 \exists \mathsf{x}_2 \ldots \exists \mathsf{x}_n \Big(\bigwedge_{1\le i<j\le n} (\mathsf{x}_i \neq \mathsf{x}_j) \wedge \bigwedge_{1\le i\le n} \mathsf{P}\mathsf{x}_i \Big)$$

$$\tau_n = \exists \mathsf{x}_1 \exists \mathsf{x}_2 \ldots \exists \mathsf{x}_n \Big(\bigwedge_{1\le i<j\le n} (\mathsf{x}_i \neq \mathsf{x}_j) \wedge \bigwedge_{1\le i\le n} \neg\mathsf{P}\mathsf{x}_i \Big)$$

Let $\Sigma = \{\sigma_n : n \ge 1\} \cup \{\tau_n : n \ge 1\}$ (so Σ says that there infinitely many elements satisfying P and infinitely many not satisfying P), and let $T = Cn(\Sigma)$. Calculate $I(T, \aleph_\alpha)$ for each ordinal α.
Hint: It may help to first think about $\alpha = 0$, $\alpha = 1$, and $\alpha = \omega$ to get the general pattern.

Problem 5: In the language $\mathcal{L} = \{\mathsf{0}, \mathsf{1}, +, \cdot, <\}$, let $\mathfrak{N}$ be the $\mathcal{L}$-structure $(\mathbb{N}, 0, 1, +, \cdot, <)$. Show that $I(Th(\mathfrak{N}), \aleph_0) = 2^{\aleph_0}$.
Hint: Use Problem 4b on Homework 7.

Challenge Problems

Problem 1: Show that there is an uncountable subset of $\mathbb{R}$ which does not have a nonempty perfect subset.

Problem 2: Let $\mathcal{U}$ be a nonprincipal ultrafilter on ω. Suppose that $\langle a_n \rangle_{n\in\omega}$ is a bounded sequence of real numbers.
a. Show that there exists a unique real number ℓ, denoted by $\mathcal{U}$-$\lim a_n$, such that for all $\varepsilon > 0$, we have

$$\{n \in \omega : |a_n - \ell| < \varepsilon\} \in \mathcal{U}$$

b. Show that if $\lim a_n = \ell$, then $\mathcal{U}$-$\lim a_n = \ell$.
It's not hard to show that $\mathcal{U}$-lim obeys the usual limit rules. Thus, a nonprincipal ultrafilter on ω gives a way to coherently define the limit of *any* bounded sequence of real numbers.