Homework 10: Due Friday, May 6

Required Problems

Problem 1: Give a careful proof (meaning that you'll use the Axiom of Choice so be explicit when using it) of the following fact: A linear ordering $(L, <)$ is a well-ordering if and only if there is no $f\colon \omega \to L$ such that $f(n+1) < f(n)$ for all $n \in \omega$.

Problem 2: Let $\mathcal{L} = \{\mathsf{R}\}$ where R is a binary relation symbol. Show that the class of well-orderings is not a weak elementary class in the language $\mathcal{L}$.

Problem 3: Over ZF, show that the statement "For all sets A and B and all surjections $f\colon A \to B$, there exists an injection $g\colon B \to A$ such that $f \circ g$ is the identity function on B" implies the Axiom of Choice.

Problem 4: The Hausdorff Maximality Principle states that every partial ordering $(P, <)$ has a maximal chain with respect to $\subseteq$ (that is, a chain H such that there is no chain I with $H \subsetneq I$). Show that the Hausdorff Maximality Principle is equivalent to the Axiom of Choice over ZF.

Problem 5:
a. Let $\mathcal{F}$ be the set of all functions from $\mathbb{R}$ to $\mathbb{R}$. Show that $|\mathcal{F}| = 2^{2^{\aleph_0}}$.
b. Let $\mathcal{C}$ be the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Show that $|\mathcal{C}| = 2^{\aleph_0}$.
Together with the fact that $2^{\aleph_0} < 2^{2^{\aleph_0}}$, this gives the worst proof ever that there is a function $f\colon \mathbb{R} \to \mathbb{R}$ that is not continuous.

Problem 6: Suppose that V is a vector space over a field F. Let B_1 and B_2 be two bases of V over F, and suppose that at least one of B_1 or B_2 is infinite. Show that $|B_1| = |B_2|$. You may use the standard linear algebra fact that if B is a finite basis of V over F and $S \subseteq V$ is finite with $|S| > |B|$, then S is linearly dependent.
Hint: Express each of the elements of B_2 as a finite linear combination of elements of B_1. How many total elements of B_1 are used in this way?