Homework 1: Due Friday, February 5

Required Problems

Problem 1: Consider the following simple generating system. Let $A = \{1, 2, 3, 4, 5, 6, 7\}$, $B = \{5\}$ and $\mathcal{H} = \{h_1, h_2\}$, where $h_1 \colon A \to A$ is given by

$$h_1(1) = 3 \quad h_1(2) = 1 \quad h_1(3) = 3 \quad h_1(4) = 7 \quad h_1(5) = 5 \quad h_1(6) = 1 \quad h_1(7) = 4$$

and $h_2 \colon A^2 \to A$ is given by

	1	2	3	4	5	6	7
1	3	6	5	6	2	2	1
2	7	2	1	1	2	1	3
3	5	6	5	1	2	2	3
4	1	4	4	4	4	4	7
5	4	5	2	5	7	3	4
6	1	7	5	1	2	1	2
7	7	6	1	5	5	1	5

Interpret the diagram as follows. If $m, n \in A$, to calculate the value of $h_2(m, n)$, go to row m and column n. For example, $h_2(1, 2) = 6$.
a. Determine V_3 and W_3 explicitly. Justify your answers.
b. Determine G explicitly. Justify your answer.

Problem 2: Let A be an arbitrary group, and let B be an arbitrary nonempty subset of A. Let $\mathcal{H} = \{h_1, h_2\}$ where $h_1 \colon A \to A$ is the inverse function and $h_2 \colon A^2 \to A$ is the group operation. Show that the simple generating system $(A, B, \mathcal{H})$ is not free.

Problem 3: Let $A = \mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$, $B_1 = \{\sqrt{2}\}$, $B_2 = \{\sqrt{2}, 16\}$, and $\mathcal{H} = \{h\}$ where $h \colon \mathbb{R}^+ \to \mathbb{R}^+$ is given by $h(x) = x^2$.
a. Describe $G(A, B_1, \mathcal{H})$ and $G(A, B_2, \mathcal{H})$ explicitly.
b. Show that $(A, B_1, \mathcal{H})$ is free, but $(A, B_2, \mathcal{H})$ is not free.
c. Define $\alpha \colon B_2 \to \mathbb{R}$ by $\alpha(\sqrt{2}) = 0$ and $\alpha(16) = \frac{7}{2}$. Define $g_h \colon \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ by letting $g_h(a, x) = \log_2 a + x$. Show that there exists a function $f \colon G(A, B_2, \mathcal{H}) \to \mathbb{R}$ with the following properties:

1. $f(b) = \alpha(b)$ for all $b \in B_2$.
2. $f(h(a)) = g_h(a, f(a))$ for all $a \in G(A, B_2, \mathcal{H})$.

Thus, we can define a function which satisfies the recursive equations in this case despite the lack of freeness.

Problem 4: Let $A = \mathbb{N}^+$, $B = \{7, 13\}$, and $\mathcal{H} = \{h_1, h_2\}$ where $h_1 \colon A \to A$ is given by $h_1(n) = 20n + 1$ and $h_2 \colon A^2 \to A$ is given by $h_2(n, m) = 2^n(2m + 1)$. Show that $(A, B, \mathcal{H})$ is free.

Problem 5: Let $\overline{\mathbb{Q}}$ be the set of all algebraic numbers. That is, $\overline{\mathbb{Q}}$ is the set of all $z \in \mathbb{C}$ such that there exists a nonzero polynomial $p(x) \in \mathbb{Q}[x]$ with rational coefficients having z as a root. Show that $\overline{\mathbb{Q}}$ is countable.

Problem 6: Let $(A, B, \mathcal{H})$ be a (not necessarily simple) generating system. Assume that B is countable, that $\mathcal{H}$ is countable (that is, there are countably many functions in $\mathcal{H}$), and that for each $h \in \mathcal{H}_k$ and $a_1, a_2, \ldots, a_k \in A$, the set $h(a_1, a_2, \ldots, a_k)$ is countable. Show that G is countable.

Challenge Problems

Problem 1: Suppose that $(A, B, \mathcal{H})$ is a simple generating system that is not free. Show that there exists a set X and functions $\alpha \colon B \to X$ and $g_h \colon (A \times X)^k \to X$ for each $h \in \mathcal{H}_k$, such that there is no function $f \colon G \to X$ satisfying the following:

1. $f(b) = \alpha(b)$ for all $b \in B$.
2. $f(h(a_1, a_2, \dots, a_k)) = g_h(a_1, f(a_1), a_2, f(a_2), \dots, a_k, f(a_k))$ for all $h \in \mathcal{H}_k$ and all $a_1, a_2, \dots, a_k \in G$.