Midterm Exam: Due Wednesday, April 13 at the Beginning of Class

- You are free to use the course textbook, the homework solutions, your personal notes, and your previous homework in solving these problems.
- You may not communicate with anybody else (student or otherwise) about the problems on the exam. You may not consult or receive assistance from any source besides those mentioned above.
- Organize your solutions and write them neatly!

Problem 1: (5 points) Let $K \prec F$ be an algebraic extension. Suppose that R is a subring of F with $K \subseteq R$. Show that R is a field.

Problem 2: (5 points) Let $\xi = e^{2\pi i/5} \in \mathbb{C}$. Let $m(x)$ be the minimal polynomial of $3 + \xi - \frac{7}{2}\xi^3$ over $\mathbb{Q}$. Prove that $m(x)$ splits in $\mathbb{Q}(\xi)$.

Problem 3: (5 points) Let $K \prec F$ be a field extension. Let $u, w \in F$ be algebraic over K. Let $g(x)$ be the minimal polynomial of u over K and let $h(x)$ be the minimal polynomial of w over K. Show that $g(x)$ is irreducible over $K(w)$ if and only if $h(x)$ is irreducible over $K(u)$.

Problem 4: (5 points) Let $K \prec F$ be a field extension with $[F : K] = 2$. Suppose that $char K = 0$. Show that $K \prec F$ is a Galois extension.

Problem 5: (5 points) Working in $\mathbb{C}$, find the splitting field of $x^4 + 5x^2 + 6$ over $\mathbb{Q}$ and compute its degree.

Problem 6: (7 points) Let $K \prec F$ be a finite extension. Suppose that $K \prec L \prec F$ and $K \prec M \prec F$. Define LM to be the smallest subfield of F containing $L \cup M$. Let $\ell = [L : K]$ and $m = [M : K]$.
a. Show that $[LM : K] \leq \ell m$.
b. Show that if $[LM : K] = \ell m$, then $L \cap M = K$.

Problem 7: (8 points) Let p be a prime and let $g(x) = x^p - x - 1 \in \mathbb{Z}/p\mathbb{Z}[x]$. Let F be a splitting field of $g(x)$ over $\mathbb{Z}/p\mathbb{Z}$. Let $u \in F$ be a root of $g(x)$. Show each of the following (in any order):
a. Show that $g(x)$ is irreducible in $\mathbb{Z}/p\mathbb{Z}[x]$.
b. Show that $u + k$ is a root of $g(x)$ for all $k \in \mathbb{Z}/p\mathbb{Z}$.
c. Show that $u^{p^p} = u$.
d. Show that $[F : \mathbb{Z}/p\mathbb{Z}] = p$.