Homework 9 : Due Wednesday, April 27

Problem 1: Let p be an odd prime and consider the group D_p.
a. Write out (with justification) the class equation of D_p.
b. Find all normal subgroups of D_p.

Problem 2: Let G be a *finite* group. Prove that G is solvable if and only if there exists a chain of subgroups

$$\{e\} = H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots \subseteq H_n = G$$

such that each H_i is a normal subgroup of H_{i+1} and each $(H_{i+1} : H_i)$ is prime.

Problem 3: Let G be a group. Given $a, b \in G$ we define the *commutator* of a and b to be $[a, b] = a^{-1}b^{-1}ab$. Let G' be the subgroup of G generated by all of the commutators, i.e.

$$G' = \langle \{[a, b] : a, b \in G\} \rangle$$

a. Show that the inverse of a commutator is a commutator.
b. Show that a conjugate of a commutator is a commutator.
c. Show that G' is a normal subgroup of G.
d. Let N be a normal subgroup of G. Show that G/N is abelian if and only if $G' \subseteq N$. Thus, G/G' is the "largest" abelian quotient of G.
e. Define a sequence $G^{(n)}$ recursively by letting $G^{(0)} = G$ and $G^{(n+1)} = (G^{(n)})'$. Thus, $G^{(1)} = G'$, $G^{(2)} = G''$, etc. Show that G is solvable if and only if there exists $n \in \mathbb{N}$ with $G^{(n)} = \{e\}$.

Problem 4: Let $f(x) \in \mathbb{Q}[x]$. Working in $\mathbb{C}$, let F be the splitting field of $f(x)$ over $\mathbb{Q}$. Suppose that $[F : \mathbb{Q}]$ is odd. Show that every root of $f(x)$ in $\mathbb{C}$ is real.

Problem 5: Let F be a finite field with $|F| = p^n$ and consider the Galois extension $\mathbb{Z}/p\mathbb{Z} \prec F$. Let $N \colon F \to \mathbb{Z}/p\mathbb{Z}$ be the norm of the extension $\mathbb{Z}/p\mathbb{Z} \prec F$ as defined in Homework 7. Let

$$d = \frac{p^n - 1}{p - 1}$$

a. Let $\sigma \colon F \to F$ be the Frobenius automorphism, i.e. $\sigma(a) = a^p$. Show that $Gal_{\mathbb{Z}/p\mathbb{Z}}F = \langle \sigma \rangle$.
b. Show that $N(a) = a^d$ for all $a \in F$.
c. Given a field K, let $K^\times = K \backslash \{0\}$ considered as a multiplicative group. Notice that $N(a) \neq 0$ for all $a \neq 0$. Letting φ be the restriction of N to $F^\times$, we know from Homework 7 that $\varphi \colon F^\times \to \mathbb{Z}/p\mathbb{Z}^\times$ is a group homomorphism (this is also immediate in this case from the formula in part b). Show that $|\ker(\varphi)| = d$ and $|\text{range}(\varphi)| = p - 1$.
d. Show that N is surjective.