Homework 7 : Due Wednesday, April 6

Problem 1:
a. Let $K \prec F$ be a finite extension with $[F : K] = n$. Suppose that $u \in F$ is such that

$$|\{\tau(u) : \tau \in Gal_K F\}| > \frac{n}{2}$$

Show that $F = K(u)$.
b. Show that $\mathbb{Q}(\sqrt[4]{2} + i) = \mathbb{Q}(\sqrt[4]{2}, i)$.

Problem 2: In this problem we show that $Gal_{\mathbb{Q}}\mathbb{R} = \{id_{\mathbb{R}}\}$. Let $\tau \in Gal_{\mathbb{Q}}\mathbb{R}$.
a. Show that $\tau(a) \geq 0$ for all $a \in \mathbb{R}$ with $a \geq 0$.
b. Show that if $a, b \in \mathbb{R}$ with $a < b$, then $\tau(a) < \tau(b)$.
c. Show that $\tau = id_{\mathbb{R}}$.
Hint for a: Think about squares.

Problem 3: Let $u = \sqrt{2 + \sqrt{2}}$.
a. Find the minimal polynomial $m(x)$ of u over $\mathbb{Q}$.
b. Working in $\mathbb{C}$, show that $\mathbb{Q}(u)$ is the splitting field of $m(x)$ over $\mathbb{Q}$.
c. Determine the structure of $Gal_{\mathbb{Q}}\mathbb{Q}(u)$, i.e. determine which well-known group it is isomorphic to.
d. Determine all intermediate fields of $\mathbb{Q}(u)$ over $\mathbb{Q}$.

Problem 4: Let $K \prec F$ be a Galois extension and let $G = Gal_K F$. Define functions $N \colon F \to F$ and $T \colon F \to F$ as follows.

$$N(a) = \prod_{\tau \in G} \tau(a) \qquad\qquad T(a) = \sum_{\tau \in G} \tau(a)$$

The function N is called the *norm* and T is called the *trace* of the extension $K \prec F$.
a. Show that $N(ab) = N(a) \cdot N(b)$ and $T(a + b) = T(a) + T(b)$ for all $a, b \in F$.
b. Show that $N(a) \in K$ and $T(a) \in K$ for all $a \in F$ (so we have $N \colon F \to K$ and $T \colon F \to K$).

Problem 5: Let $K \prec F$ be a Galois extension and let $G = Gal_K F$. Suppose that L and M are intermediate fields, i.e. $K \prec L \prec F$ and $K \prec M \prec F$. Show that the following are equivalent.
1) There exists an isomorphism $\alpha \colon L \to M$ with $\alpha(k) = k$ for all $k \in K$.
2) There exists $\tau \in G$ with $\tau(L) = M$.
3) The subgroups L' and M' of G are conjugate in G, i.e. there exists $\tau \in G$ with $\tau L' \tau^{-1} = M'$.