Homework 5 : Due Wednesday, March 2

Problem 1: Prove that $\mathbb{Q}(\sqrt{2}+\sqrt{3})$ is a normal extension of $\mathbb{Q}$.

Problem 2: Working in $\mathbb{C}$, find the splitting field over $\mathbb{Q}$ of each of the following and compute its degree.
a. $x^4 - 2$
b. $x^4 + x^2 + 1$
c. $x^6 - 4$

Problem 3: Let $p(x) = x^3 - 3x^2 + 6x + 1 \in \mathbb{Q}[x]$.
a. Show that $p(x)$ has a unique real root.
b. Working in $\mathbb{C}$, show that the splitting field of $p(x)$ over $\mathbb{Q}$ has degree 6.

Problem 4: Suppose that $\mathbb{Q} \prec F \prec \mathbb{C}$ and $[F : \mathbb{Q}] = 2$.
a. Show that there exists $r \in \mathbb{Q}$ with $F = \mathbb{Q}(\sqrt{r})$.
b. Let $a, b \in \mathbb{Z}$ with $b > 0$. Show that $\mathbb{Q}(\sqrt{\frac{a}{b}}) = \mathbb{Q}(\sqrt{ab})$.
c. A nonzero integer d is *squarefree* if it is not divisible by p^2 for any prime p. Show that there exists a squarefree $d \in \mathbb{Z}$ with $F = \mathbb{Q}(\sqrt{d})$.

Problem 5: Suppose that $K \prec F$ is a finite extension. Let $g(x) \in K[x]$ be irreducible and suppose that $\deg(g(x)) = p$ a prime. Show that if $g(x)$ is not irreducible in $F[x]$, then $p \mid [F : K]$.
Hint: First extend F to a field L where $g(x)$ has a root.