Homework 4 : Due Wednesday, February 23

Problem 1: Let $K \prec F$ and $L \prec E$ be finite extensions. Let $\alpha\colon K \to L$ be an isomorphism and let $\tau\colon F \to E$ be an isomorphism which is an extension of α, i.e. $\tau(a) = \alpha(a)$ for all $a \in K$. Show that $[F : K] = [E : L]$.

Problem 2:
a. Show that $\sqrt{3} \notin \mathbb{Q}(\sqrt{2})$.
b. Show that $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}] = 4$.
c. Show that $\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(\sqrt{2} + \sqrt{3})$.
d. Find a fourth degree monic polynomial $p(x) \in \mathbb{Q}[x]$ that has $\sqrt{2} + \sqrt{3}$ as a root.
e. Use parts b, c, and d to conclude that $p(x)$ is the minimal polynomial of $\sqrt{2} + \sqrt{3}$ over $\mathbb{Q}$.

Problem 3: Suppose that $K \prec F$. Let $u \in F$ be algebraic over K.
a. Give an example of the above situation where $K(u) \neq K(u^2)$.
b. Suppose that the minimal polynomial of u over K has odd degree. Show that $K(u) = K(u^2)$.

Problem 4:
a. Show that $\mathbb{Q}(e^{2\pi i/3}) = \mathbb{Q}(i\sqrt{3})$.
b. Find $[\mathbb{Q}(e^{2\pi i/11}, \sqrt[3]{7}) : \mathbb{Q}]$.
c. Find $[\mathbb{Q}(31 + 7\sqrt[5]{2} - 13\sqrt[5]{8} + 42\sqrt[5]{16}) : \mathbb{Q}]$.

Problem 5: Let $a_1, a_2, \dots, a_n \in \mathbb{Q}$ with each $a_i > 0$. Show that $\sqrt[3]{2} \notin \mathbb{Q}(\sqrt{a_1}, \sqrt{a_2}, \dots, \sqrt{a_n})$.

Problem 6: Let $K = \mathbb{Z}/2\mathbb{Z}$ and let $p(x) = x^3 + x + 1 \in K[x]$. Notice that $p(x)$ is irreducible in $K[x]$ because it has degree 3 and has no roots in K. In class, we showed how to construct a extension of K in which $p(x)$ has a root by considering the field

$$F = K[x]/\langle x^3 + x + 1 \rangle$$

If we let $u = \overline{x}$, then we can write

$$F = \{a + bu + cu^2 : a, b, c \in \mathbb{Z}/2\mathbb{Z}\}$$

where we add in the obvious way and multiply using the fact that $u^3 + u + 1 = 0$ and hence $u^3 = -u - 1 = u + 1$.
a. Write out an 8×8 tables giving addition and multiplication in F.
b. Factor the polynomial $x^3 + x + 1$ into irreducibles in $F[x]$.
c. Find the minimal polynomial of $u + 1 \in F$ over K.