Homework 3 : Due Wednesday, February 16

Problem 1: Let V and W be vector spaces over a field F and assume that $\dim_F V < \infty$. Let $T\colon V \to W$ be a linear transformation (i.e. $T(u+v) = T(u) + T(v)$ and $T(\lambda u) = \lambda T(u)$ for all $u, v \in V$ and $\lambda \in F$).
a. Show that $\ker(T)$ is a subspace of V and $\operatorname{range}(T)$ is a subspace of W.
b. Show that $\dim_F V = \dim_F \ker(T) + \dim_F \operatorname{range}(T)$.

Problem 2: Let K and F be fields with $K \prec F$.
a. Show that $K = F$ if and only if $[F : K] = 1$.
b. Suppose that $f(x), g(x) \in K[x]$ and that $f(x) \mid g(x)$ in $F[x]$. Show that $f(x) \mid g(x)$ in $K[x]$.

Problem 3: Although we have not worked through all of the details, the fundamental results of linear algebra work over any field. In particular, given an $n \times n$ matrix A over a field F, the following are equivalent:

- A is invertible.
- The columns of A are linearly independent in F^n.
- The columns of A span F^n.

Use this to determine the number of invertible $n \times n$ matrices over $F = \mathbb{Z}/p\mathbb{Z}$ for a prime p.

Problem 4: Find the minimal polynomial of each of the following over $\mathbb{Q}$.
a. $\sqrt[3]{2} + 1$
b. $\sqrt{2} + i$
c. $e^{2\pi i/8} = \cos(\pi/4) + i\sin(\pi/4) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$

Problem 5: Let $f(x) = x^4 + 6x - 2 \in \mathbb{Q}[x]$. Let u be some (any) root of $f(x)$ in $\mathbb{C}$.
a. Show that $f(x)$ is irreducible in $\mathbb{Q}[x]$.
b. We know that $\mathbb{Q}(u) = \{a + bu + cu^2 + du^3 : a, b, c, d \in \mathbb{Q}\}$. Find values of a, b, c, d for $u^6 - 2u^3$ and $1/u$.

Problem 6: Suppose that $F \subseteq R$ where R is an integral domain and F is a field which is a subring of R.
a. Give an example of the above situation where R is not a field.
b. In this situation, we can still view R as a vector space over F. Show that if $\dim_F R < \infty$, then R is a field.
Hint: Fix $a \in R$ with $a \neq 0$. To find an inverse for a, define a certain linear transformation on R using a and think about the range.