Homework 2 : Due Wednesday, February 9

Problem 1: Let F be a field. Define a function $D\colon F[x] \to F[x]$ by letting

$$\begin{aligned} D(\sum_{k=0}^{n} a_k x^k) &= \sum_{k=1}^{n} k a_k x^{k-1} \\ &= \sum_{k=0}^{n-1} (k+1) a_{k+1} x^k \end{aligned}$$

In other words, $D(f(x))$ is the "formal derivative" of $f(x)$. In the above equation, you should interpret the "k" in $ka_k x^{k-1}$ as $1 + 1 + \cdots + 1$ (where there are k ones in the sum). For example, in $\mathbb{Z}/2\mathbb{Z}[x]$ we have

$$D(x^7 + x^4 + x^2 + x + 1) = x^6 + 1$$

because $7 = 1$, $4 = 0$, and $2 = 0$ in $\mathbb{Z}/2\mathbb{Z}$.
a. Show that $D(f(x) + g(x)) = D(f(x)) + D(g(x))$ for all $f(x), g(x) \in F[x]$.
b. Show that $D(f(x) \cdot g(x)) = D(f(x)) \cdot g(x) + f(x) \cdot D(g(x))$ for all $f(x), g(x) \in F[x]$.

Problem 2: Consider $\mathbb{C}$ as a field.
a. Show that $\psi(a + bi) = a - bi$ is an automorphism (so you should show it preserves both addition and multiplication).
b. Suppose that $\varphi\colon \mathbb{C} \to \mathbb{C}$ is an automorphism with the property that $\varphi(a) = a$ for all $a \in \mathbb{R}$. Show that either $\varphi = \psi$ or $\varphi = id_{\mathbb{C}}$.
Cultural Aside: It turns out that there are infinitely many automorphisms of $\mathbb{C}$ (in fact uncountably many), but in a certain precise sense it is impossible to "construct" an automorphism other than the above two.

Problem 3: Let R be a commutative ring (with identity). An element $a \in R$ is *nilpotent* if there exists $n \in \mathbb{N}^+$ with $a^n = 0$.
a. Let I be the set of nilpotent elements of R. Show that I is an ideal of R.
b. Show that R/I has no nonzero nilpotent elements.

Problem 4: Let $R = C[0, 1]$ be the ring of all continuous functions $f\colon [0, 1] \to \mathbb{R}$ with addition and multiplication defined as pointwise addition and multiplication of functions.
a. Let $c \in [0, 1]$. Show that $M_c = \{f \in R : f(c) = 0\}$ is a maximal ideal of R.
b. Let $c \in [0, 1]$. Let $p \in R$ be the function $p(x) = x - c$. Show that $M_c \neq \langle p \rangle$.
c.* Show that if M is a maximal ideal of R, then $M = M_c$ for some $c \in [0, 1]$.
Note: Part c will not be graded since it requires some nontrivial analysis (which is not a prerequisite for the course). But it's a really nice problem and I encourage you to try it.

Problem 5: Find all values of $n \in \mathbb{Z}$ such that $x^3 + nx + 2$ is irreducible in $\mathbb{Q}[x]$.

Problem 6: Factor the polynomials $x^6 - 1$ and $x^8 - 1$ into irreducibles in each of $\mathbb{Q}[x]$, $\mathbb{Z}/2\mathbb{Z}[x]$ and $\mathbb{Z}/3\mathbb{Z}[x]$. Explain why your factors are irreducible.

Problem 7: Show that $f(x) = (x - 1)(x - 2) \cdots (x - n) - 1$ is irreducible in $\mathbb{Q}[x]$ for each $n \in \mathbb{N}^+$.