Homework 10 : Due Monday, May 9

Notation: Throughout this assignment, let $\xi_n = e^{2\pi i/n} \in \mathbb{C}$.

Problem 1: Let $\mathbb{Q} \prec F$ be a Galois extension and suppose that $Gal_{\mathbb{Q}}F \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Show that there exist squarefree integers $c, d \in \mathbb{Z}$ such that $F = \mathbb{Q}(\sqrt{c}, \sqrt{d})$.

Problem 2: Let K be a field of prime characteristic p. Let F be a splitting field of $f(x) = x^n - 1$ over K. Write $n = p^k m$ where $p \nmid m$. Show that $f(x)$ has exactly m distinct roots in F.

Problem 3:
a. Show that $\Phi_n(0) = 1$ for all $n \geq 2$.
b. Let p be prime. Find an explicit formula for $\Phi_{p^2}(x)$.
c. Suppose that n is odd. Show that $\xi_{2n} \in \mathbb{Q}(\xi_n)$ by finding an explicit formula for ξ_{2n} in terms of ξ_n. Conclude that $\mathbb{Q}(\xi_n) = \mathbb{Q}(\xi_{2n})$.
d. Suppose that $n \geq 3$ is odd. Show that $\Phi_{2n}(x) = \Phi_n(-x)$.

Problem 4: Let p be an odd prime. In this problem, we show that the only degree two extension of $\mathbb{Q}$ inside $\mathbb{Q}(\xi_p)$ is:

$$\begin{cases} \mathbb{Q}(\sqrt{p}) & \text{if } p \equiv_4 1 \\ \mathbb{Q}(i\sqrt{p}) & \text{if } p \equiv_4 3 \end{cases}$$

Letting $m = (-1)^{(p-1)/2}$, we have $m = 1$ if $p \equiv_4 1$ and $m = -1$ if $p \equiv_4 3$. With this notation, we can consolidate the above two calling them both $\mathbb{Q}(\sqrt{mp})$.
a. Show that $\mathbb{Q}(\xi_p)$ has a unique subfield E with $[E : \mathbb{Q}] = 2$.
b. Show that $(1-\xi)(1-\xi^2)\cdots(1-\xi^{p-1}) = p$. (*Hint:* Think about $\Phi_p(x)$)
c. Show that there exists $k \in \mathbb{Z}$ with $(1-\xi)(1-\xi^2)\cdots(1-\xi^{p-1}) = m\xi^k[(1-\xi)(1-\xi^2)\cdots(1-\xi^{(p-1)/2})]^2$.
d. Show that there exists $u \in \mathbb{Q}(\xi)$ with $u^2 = mp$.
e. Show that $\mathbb{Q}(\sqrt{mp})$ is the unique field E from part a.

Problem 5: Show that $\sqrt[3]{2} \notin \mathbb{Q}(\xi_n)$ for all $n \in \mathbb{N}^+$.