Homework 1 : Due Wednesday, February 2

Problem 1: Let G be a finite group of order n. Let $k \in \mathbb{Z}$ be such that $\gcd(k, n) = 1$. Define a function $\varphi_k \colon G \to G$ by letting $\varphi_k(a) = a^k$.
a. Give an example to show that φ_k need not be an automorphism of G.
b. Show that φ_k is always a bijection.
Hint: We proved part b in class for cyclic groups. You can leverage that even if G itself is not cyclic.

Problem 2: Let G be a group and let $g \in G$. Define a function $\psi_g \colon G \to G$ by letting $\psi_g(a) = gag^{-1}$. A function of the form ψ_g is called an *inner automorphism* of G.
a. Show that ψ_g is an automorphism of G for each $g \in G$ (apropos of the name).
b. Show that the function $f \colon G \to Aut(G)$ defined by $f(g) = \psi_g$ is a homomorphism.
c. Show that $Inn(G)$, the set of inner automorphisms of G, is a normal subgroup of $Aut(G)$.
d. Show that $G/Z(G) \cong Inn(G)$.

Problem 3:
a. Let G and H be groups and let $\varphi \colon G \to H$ and $\psi \colon G \to H$ be homomorphisms. Show that

$$\{g \in G : \varphi(g) = \psi(g)\}$$

is a subgroup of G. Conclude that if $G = \langle X \rangle$ and $\varphi(a) = \psi(a)$ for all $a \in X$, then $\varphi = \psi$.
b. Show that $Aut(S_3) \cong S_3$. *Hint:* Start with Problem 2.
c. Let D_4 be the set of symmetries of a square. Show that $|Aut(D_4)| \leq 8$.
Cultural Aside: It's a curious fact that $Aut(S_n) \cong S_n$ for all $n \geq 3$ except for the lone case when $n = 6$.

Problem 4: A subgroup M of a group G is *maximal* if $M \neq G$ and there is no subgroup H of G with $M \subsetneq H \subsetneq G$. Show that if G is a finite group with exactly one maximal subgroup, then G is cyclic of prime power order.

Problem 5: Show that an infinite group has infinitely many subgroups.

Problem 6: Let G be a finite group with $|G| = p^a n$ where p is prime and $p \nmid n$. Let P be a subgroup of G with $|P| = p^a$. Suppose that N is a normal subgroup of G and that $|N| = p^b k$ where $p \nmid k$. Show that $|N \cap P| = p^b$.
Aside: If you know the jargon (which is not required here), this problem is asking you to show that the intersection of a Sylow p-subgroup of G with a normal subgroup N of G results in a Sylow p-subgroup of N.