Homework 8 : Due Wednesday, April 18

Problem 1: Let $K = \mathbb{Q}(\sqrt{10})$.
a. Show that $3 + \sqrt{10}$ is the fundamental unit in $\mathcal{O}_K$.
b. Show that $4 - \sqrt{10}$ and $98 + 31\sqrt{10}$ are associates in $\mathcal{O}_K$.

Problem 2:
a. Let $\alpha \in \mathbb{C}$ be algebraic over $\mathbb{Q}$. Show that there exists $m \in \mathbb{Z}\backslash\{0\}$ such that $m\alpha$ is an algebraic integer.
b. Let K be a number field. Show that K is the field of fractions of $\mathcal{O}_K$, i.e. show that K is smallest subfield of $\mathbb{C}$ containing the ring $\mathcal{O}_K$.

Problem 3: Let $d \in \mathbb{Z}$ be square-free with $d \geq 2$. Show that $\mathbb{Z}[\sqrt{d}]$ is dense in the real line. That is, show that given any $r, s \in \mathbb{R}$ with $r < s$, there exists $\alpha \in \mathbb{Z}[\sqrt{d}]$ with $r < \alpha < s$.
Note: Since $\mathbb{Z}[\sqrt{d}] \subseteq \mathcal{O}_{\mathbb{Q}(\sqrt{d})}$, if follows that $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ is dense in the real line when $d \equiv 1 \pmod 4$.

Problem 4: Let $d \in \mathbb{Z}$ be square-free with $d \leq -3$. In this problem we work in $\mathbb{Z}[\sqrt{d}]$ (so if $d \equiv 1 \pmod 4$, then we are *not* working in the ring of algebraic integers).
a. Show that $\sqrt{d}$ and $1 + \sqrt{d}$ are both irreducible in $\mathbb{Z}[\sqrt{d}]$.
b. Show that at least one of $\sqrt{d}$ or $1 + \sqrt{d}$ is not prime in $\mathbb{Z}[\sqrt{d}]$.
c. Show that $\mathbb{Z}[\sqrt{d}]$ is not a UFD.

Problem 5: Let $K = \mathbb{Q}(\sqrt{-5})$. Working in $R = \mathcal{O}_K = \mathbb{Z}[\sqrt{-5}]$, let $I = \langle 2, 1 + \sqrt{-5} \rangle$.
a. Show that I is a nonprincipal ideal of R.
b. Show that $|R/I| = 2$.
c. Show that I is a maximal (and hence prime) ideal of R.