Homework 7 : Due Friday, March 16

Problem 1: Find the minimal polynomial of each of the following over $\mathbb{Q}$.
a. $\sqrt[3]{2}+1$
b. $\sqrt{2-\sqrt{2}}$
c. $\sqrt{3-2\sqrt{2}}$.

Problem 2:
a. Show that $x^5+x^2+\overline{1} \in (\mathbb{Z}/2\mathbb{Z})[x]$ is irreducible in $(\mathbb{Z}/2\mathbb{Z})[x]$.
b. Show that $3x^5+10x^4-x^2+5$ is irreducible in $\mathbb{Q}[x]$.

Problem 3: Let $p(x)=x^3+9x+6 \in \mathbb{Q}[x]$.
a. Show that $p(x)$ is an irreducible polynomial in $\mathbb{Q}[x]$ with a unique real root.
b. Let α be any root of $p(x)$. We know that

$$\mathbb{Q}(\alpha)=\{a+b\alpha+c\alpha^2 : a,b,c \in \mathbb{Q}\}$$

Find the multiplicative inverse of $1+\alpha \in \mathbb{Q}(\alpha)$ and write it in the form $a+b\alpha+c\alpha^2$ where $a,b,c \in \mathbb{Q}$.

Problem 4: Let $\pi \in \mathbb{Z}[i]$ be prime and let $I=\langle \pi \rangle$. Show that $\alpha^{N(\pi)}+I=\alpha+I$ for all $\alpha \in \mathbb{Z}[i]$.
Hint: This should resemble an important fact about $\mathbb{Z}$.

Problem 5: Let R be a UFD with finitely many units. Show that every nonzero element r has finitely many divisors in R, and give a formula for the number of such divisors based on a factorization of r into irreducibles.

Problem 6: Determine all $(x,y) \in \mathbb{Z}^2$ satisfying $x^3=y^2+4$.
Hint: Break this up into cases based on whether y is even or odd. When y is even, make use of Problem 5 on Homework 6.