Midterm Exam: Due Wednesday, April 14 at the Beginning of Class

- You are free to use the two course books, the notes, the homework solutions, your notes, and your previous homework in solving these problems.
- You may not talk to other students or discuss the problems with any other people. You may not consult other books or online resources.
- Organize your solutions and write them neatly!

Problem 1: (5 points) Show that $168 \mid (x^6 - 1)$ whenever $\gcd(x, 42) = 1$.

Problem 2: (5 points) Let p be an odd prime. Show that if a and b are primitive roots modulo p, then ab is not a primitive root modulo p.

Problem 3: (5 points) Suppose that p is a prime and $p \equiv_4 3$. Suppose that a is a quadratic residue modulo p. Let $b = a^{\frac{p+1}{4}}$. Show that $b^2 \equiv_p a$. (This gives a "constructive" way to find square roots modulo these primes).

Problem 4: (5 points) Show that $8 + i$ divides $7^{96} - 1$ in $\mathbb{Z}[i]$

Problem 5: (7 points) Suppose that $m, k \in \mathbb{Z}$ where $m \geq 2$ and $k \geq 1$. Suppose further that m is squarefree (i.e. that $p^2 \nmid m$ for all primes p). Define $f \colon \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ by letting $f(x) = x^k$. Show that f is a bijection if and only if $\gcd(k, \varphi(m)) = 1$.

Problem 6: (7 points)
a. (2 points) Give an example of a PID R together with a nonzero element $a \in R$ such that a has infinitely many divisors.
b. (5 points) Suppose that R is a PID with finitely many units. Show that every nonzero element $a \in R$ has finitely many divisors, and give a formula for the number of such divisors.

Problem 7: (6 points) Let $n \in \mathbb{N}^+$. We say that n is the sum of consecutive positive integers if there exist $k, m \in \mathbb{N}^+$ with

$$n = m + (m + 1) + (m + 2) + \cdots + (m + k)$$

Prove that $n \in \mathbb{N}^+$ is the sum of consecutive positive integers if and only if n is not a power of 2.