Homework 9 : Due Wednesday, April 28

Problem 1: Let $c, d \in \mathbb{Z}$ be squarefree with $c \neq d$. Show that $\mathbb{Q}(\sqrt{c}) \cap \mathbb{Q}(\sqrt{d}) = \mathbb{Q}$.

Problem 2: Let $K = \mathbb{Q}(\sqrt{10})$.
a. Show that the fundamental unit in $\mathcal{O}_K$ is $3 + \sqrt{10}$.
b. Show that $4 - \sqrt{10}$ and $98 + 31\sqrt{10}$ are associates in $\mathcal{O}_K$.

Problem 3: Suppose that $d \in \mathbb{N}^+$ is squarefree and that d has a prime divisor p with $p \equiv_4 3$. Show that every element of $U(\mathcal{O}_{\mathbb{Q}(\sqrt{d})})$ has norm 1.
Hint: First consider the case where $d = p$.

Problem 4: Let R be the subring of $\mathbb{Q}$ consisting all rational numbers which can be written with an odd denominator, i.e.

$$R = \{\frac{a}{b} : a, b \in \mathbb{Z} \text{ and } b \text{ is odd}\}$$

a. Show 2 is prime in R.
b. Show that every prime in R is an associate of 2 (so R has a unique prime up to associates).
c. Show that R is a UFD.

Problem 5: Let $K = \mathbb{Q}(\sqrt{-5})$ and let $R = \mathcal{O}_K$. Let $I = \langle 2, 1 + \sqrt{-5} \rangle$ and let $J = \langle 2 \rangle$.
a. Show that $|R/J| = 4$ and give addition and multiplication tables for R/J.
b. Show that I is a nonprincipal ideal of R.
c. Show that $I^2 = J$ (where $I^2 = II$ is the product of the ideal I with itself as defined in Homework 6).
d. Show that $|R/I| = 2$.
e. Show that I is a maximal (and hence prime) ideal of R.