Homework 8 : Due Wednesday, April 21

Problem 1: Find (with proof) the minimal polynomial of $\sqrt{2-\sqrt{2}}$ over $\mathbb{Q}$.

Problem 2: Find values of $a, b \in \mathbb{N}^+$ such that $[\mathbb{Q}(\sqrt[3]{a+\sqrt[4]{b}}) : \mathbb{Q}] = 12$. Justify your answer.

Problem 3:
a. Show that $x^5 + x^2 + 1 \in (\mathbb{Z}/2\mathbb{Z})[x]$ is irreducible in $(\mathbb{Z}/2\mathbb{Z})[x]$.
b. Show that $3x^5 + 2x^4 - x^2 + 5$ is irreducible in $\mathbb{Q}[x]$.

Problem 4:
a. Show that $\mathbb{Q}(\sqrt{2}+\sqrt{3}) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$.
b. Show that $x^4 - 10x^2 + 1$ is irreducible in $\mathbb{Q}[x]$.
Hint: It is possible to do the parts of this problem in either order.

Problem 5: Let $p(x) = x^3 + 3x + 2$.
a. Show that $p(x)$ is irreducible in $\mathbb{Q}[x]$.
b. Show that $p(x)$ has exactly one root in $\mathbb{R}$.
c. Let α be the unique real root of $p(x)$. We know from part a that $p(x)$ is the minimal polynomial of α over $\mathbb{R}$, hence $\{1, \alpha, \alpha^2\}$ is a basis of $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$ by Theorem 6.14. In particular, we have

$$\mathbb{Q}(\alpha) = \{a + b\alpha + c\alpha^2 : a, b, c \in \mathbb{Q}\}$$

Now we clearly have $\alpha^5 - \frac{2}{5}\alpha^3 + 42 \in \mathbb{Q}(\alpha)$. Find $a, b, c \in \mathbb{Q}$ such that

$$\alpha^5 - \frac{2}{5}\alpha^3 + 42 = a + b\alpha + c\alpha^2$$