Homework 7 : Due Wednesday, March 17

Problem 1: Let R be an integral domain. Some books require a Euclidean function N on R to also satisfy $N(ab) \geq N(a)$ whenever $a, b \in R$ and $b \neq 0$ (in other words, nonzero multiples of an element always have larger "size"). Notice that the standard Euclidean functions on $\mathbb{Z}$, $F[x]$, and $\mathbb{Z}[i]$ do indeed satisfy this. Suppose that you have a Euclidean function N on R with this additional property.
a. Show that $u \in R$ is a unit if and only if $N(u) = N(1)$.
b. Show that if $a, b \in R$ are associates then $N(a) = N(b)$.
c. Find a counterexample to the converse of part b.
Note: It turns out that if an integral domain R has a Euclidean function in our weaker sense, then it must also have a Euclidean function (perhaps different) with this additional property.

Problem 2: Factor each of the following elements of $\mathbb{Z}[i]$ into primes. Justify why your final factors are indeed prime in $\mathbb{Z}[i]$.
a. 29
b. 252
c. $33 + 22i$
d. $5 + 7i$

Problem 3: Suppose that R is a Euclidean domain with Euclidean function $N \colon R\backslash\{0\} \to \mathbb{N}$.
a. Suppose that $a, b \in R$ with $b \neq 0$. Suppose that $a = qb + r$ where either $r = 0$ or $N(r) < N(b)$. Show that d is a greatest common divisor of a and b if and only if d is a greatest common divisor of b and r. Therefore, you can use the Euclidean Algorithm to find a greatest common divisor in any Euclidean domain (hence the name).
b. Use the Euclidean Algorithm to find all greatest common divisors of $8 + 9i$ and $10 - 5i$ in $\mathbb{Z}[i]$. Justify your steps.

Problem 4:
a. Show how to write 108,290 as the sum of two squares by first factoring the number and then working your way up to a solution. (In other words, saying that you found a solution by exhaustive search won't suffice).
b. Prove that if an integer is the sum of two rational squares then it is the sum of two integer squares (for example, $13 = (1/5)^2 + (18/5)^2 = 2^2 + 3^2$)

Problem 5: Show that $\mathbb{Z}[x]$, the ring of polynomials over the integers, is not a Euclidean domain.

Problem 6: Show that $\mathbb{Z}[i]/I$ is finite for all nonzero ideals I of $\mathbb{Z}[i]$.