Homework 11 : Due Wednesday, May 12

Problem 1: Let $K = \mathbb{Q}(\sqrt[4]{2})$.
a. Describe the four emeddings of K into $\mathbb{C}$.
b. Show that these four embeddings send $\sqrt[4]{2} + \sqrt{2} + \sqrt[4]{8}$ to four distinct elements.
c. Show how to conclude that $K = \mathbb{Q}(\sqrt[4]{2} + \sqrt{2} + \sqrt[4]{8})$ using part b.

Problem 2: Suppose that $\alpha \in \mathbb{C}$ is algebraic over $\mathbb{Q}$. Show that there exists a nonzero $k \in \mathbb{Z}$ such that $k\alpha$ is an algebraic integer.

Problem 3: Let $R = \mathcal{O}_{\mathbb{Q}(\sqrt{-2})}$. Suppose that $p \in \mathbb{Z}$ is a prime with either $p \equiv_8 5$ or $p \equiv_8 7$. Show that $R/\langle p \rangle$ is a field with p^2 elements.

Problem 4: Let $R = \mathbb{Z}[\frac{1+\sqrt{-5}}{2}]$ be the smallest subring of $\mathbb{C}$ containing $\mathbb{Z} \cup \{\frac{1+\sqrt{-5}}{2}\}$. Notice that $-5 \not\equiv_4 1$, so this is *not* the ring of integers in $\mathbb{Q}(\sqrt{-5})$.
a. Show that $1/2^n \in R$ for all $n \in \mathbb{N}^+$.
b. Show directly from part a (i.e. not using our theory on algebraic integers) that the set R is not finitely generated as an additive abelian group. That is, show that there does not exist $a_1, a_2, \dots, a_n \in R$ such that

$$R = \{k_1a_1 + k_2a_2 + \cdots + k_na_n : k_i \in \mathbb{Z}\}$$

Note: We will see why the results of this problem imply that the ring R is "too large" to be rightfully considered as the ring of algebraic integers in $\mathcal{O}_{\mathbb{Q}(\sqrt{-5})}$.

Problem 5: Let $R = \mathbb{Z}[\sqrt{-3}]$ be the smallest subring of $\mathbb{C}$ containing $\mathbb{Z} \cup \{\sqrt{-3}\}$. Notice that $-3 \equiv_4 1$, so this is *not* the ring of integers in $\mathbb{Q}(\sqrt{-3})$. Let $I = \langle 2, 1 + \sqrt{-3} \rangle$ and $J = \langle 2 \rangle$.
a. Show that $J \subsetneq I \subsetneq R$.
b. Show that J is not a prime ideal of R.
c. Show that I is the unique prime ideal of R containing J.
d. Show that $I^2 \subsetneq J$.
Note: You should contrast this problem with Problem 5 on Homework 9. We will see why the results in this problem imply that the ring R is "too small" to be rightfully considered as the ring of algebraic integers in $\mathcal{O}_{\mathbb{Q}(\sqrt{-3})}$.