Homework 10 : Due Wednesday, May 5

Problem 1: Suppose that $\alpha \in \mathbb{C}$ is algebraic over $\mathbb{Q}$ and that $[\mathbb{Q}(\alpha) : \mathbb{Q}]$ is odd. Show that $\mathbb{Q}(\alpha) = \mathbb{Q}(\alpha^2)$.

Problem 2: Suppose that $\alpha \in \mathbb{R}$ is such that $\sin\alpha$ is algebraic over $\mathbb{Q}$.
a. Show that $\cos\alpha$ is algebraic over $\mathbb{Q}$.
b. Show that $\sin(\alpha/2)$ is algebraic over $\mathbb{Q}$.
c. Show that $\sin 3^\circ$ is algebraic over $\mathbb{Q}$.

Problem 3: Let $R = \mathcal{O}_{\mathbb{Q}(\sqrt{-5})}$. We know that R is not a PID and in fact you showed in the last homework that $\langle 2, 1+\sqrt{-5}\rangle$ is nonprincipal. Thus, there is no guarantee that greatest common divisors always exist in R.
a. Show that 2 and $1+\sqrt{-5}$ do in fact have a greatest common divisor in R.
b. Show that 6 and $2+2\sqrt{-5}$ have no greatest common divisor in R.

Problem 4: Letting $R = \mathbb{Z}[i] = \mathcal{O}_{\mathbb{Q}(i)}$, we proved that an odd prime $p \in \mathbb{Z}$ was irreducible in R if and only if $p \equiv_4 3$, which is if and only if $\left(\frac{-1}{p}\right) = -1$. Prove the analogue for -2. That is, letting $R = \mathcal{O}_{\mathbb{Q}(\sqrt{-2})}$, show that an odd prime $p \in \mathbb{Z}$ is irreducible in R if and only if $\left(\frac{-2}{p}\right) = -1$ (which by Corollary 4.10 is if and only if $p \equiv_8 5$ or $p \equiv_8 7$).
Hint: Try to follow our proof for the Gaussian Integers as much as possible. One direction will make essential use of the fact that R is a PID. You should point out where.

Problem 5: Let $p \in \mathbb{N}^+$ be prime. The polynomial $x^p - 1 \in \mathbb{Q}[x]$ factors as

$$x^p - 1 = (x-1)(x^{p-1} + x^{p-2} + \cdots + x + 1)$$

Show that $f(x) = x^{p-1} + x^{p-2} + \cdots + x + 1$ is irreducible over $\mathbb{Q}$ (it follows that $f(x)$ is minimal polynomial of $e^{2\pi i/p}$ over $\mathbb{Q}$).
Hint: Make use of the following clever trick. It suffices to show that $f(x+1)$ is irreducible because if $f(x) = g(x)h(x)$ with $g(x)$ and $h(x)$ nonconstants, then $f(x+1) = g(x+1)h(x+1)$ and both $g(x+1)$ and $h(x+1)$ remain nonconstant. You can get a nice description of $f(x+1)$ using the factorization $x^p - 1 = (x-1)f(x)$.