Homework 1 : Due Wednesday, February 3

Problem 1: Show that $4 \nmid n^2 + 2$ for all $n \in \mathbb{Z}$.

Problem 2: Prove that there are no integers x and y satisfying $3y^2 - 1 = x^2$.

Problem 3: Suppose that $a, b \in \mathbb{Z}$ are relatively prime and that both $a \mid n$ and $b \mid n$.
a. Without using the Fundamental Theorem of Arithmetic, show that $ab \mid n$.
b. Using the Fundamental Theorem of Arithmetic, show that $ab \mid n$.

Problem 4: Suppose that $a, b \in \mathbb{N}$ are relatively prime and that ab is a square. Show that each of a and b are squares. Is the assumption that a and b are relatively prime necessary?

Problem 5: Find, with justification, the smallest integer $n \geq 1$ such that $n/2$ is a square, $n/3$ is a cube, and $n/5$ is a fifth power.

Problem 6: Show that there are arbitrarily large gaps in the primes, i.e. that for any $n \in \mathbb{N}$, there exists n consecutive composite numbers.
Hint: $n!$ is your friend.

Problem 7: Prove that there are infinitely many primes of the form $6n + 5$.
Hint: Look at Lemma 5.2 and Theorem 5.3 in Bolker. This is one very special case of a famous theorem of number theory known as *Dirichlet's Theorem*, which says that if a and b are relatively prime, then there are infinitely many primes of the form $an + b$.

Problem 8: Fix a prime p. Define a function $e_p \colon \mathbb{Z} \to \mathbb{N} \cup \{\infty\}$ as follows. Let $e_p(0) = \infty$, and given $a \in \mathbb{Z} - \{0\}$, let $e_p(a)$ be the largest $k \in \mathbb{N}$ such that $p^k \mid a$.
a. Show that $e_p(ab) = e_p(a) + e_p(b)$ for all $a, b \in \mathbb{Z}$.
b. Show that $e_p(a + b) \geq \min\{e_p(a), e_p(b)\}$ for all $a, b \in \mathbb{Z}$.

Problem 9: Fix a prime p. We extend the function $e_p \colon \mathbb{Z} \to \mathbb{N} \cup \{\infty\}$ to a function $v_p \colon \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$ by letting $v_p(0) = \infty$ and $v_p(\frac{a}{b}) = e_p(a) - e_p(b)$.
a. Show that this function is well-defined, i.e. if $a, b, c, d \in \mathbb{Z}$ with $\frac{a}{b} = \frac{c}{d}$, then $v_p(\frac{a}{b}) = v_p(\frac{c}{d})$.
b. Show that $v_p(xy) = v_p(x) + v_p(y)$ for all $x, y \in \mathbb{Q}$.
c. Show that $v_p(x + y) \geq \min\{v_p(x), v_p(y)\}$ for all $x, y \in \mathbb{Q}$.
I suggest using the properties of e_p you established in Problem 8 rather that doing everything from scratch. This problem shows that v_p is a special type of function called a *discrete valuation* on the field $\mathbb{Q}$. It is called the p-adic valuation on $\mathbb{Q}$.