Homework 8: Due Friday, May 14

Exercises

Section 4.3: #9, 12.

Section 4.4: #1, 16.

Section 4.5: #6, 12.

Section 4.8: #1, 4.

Section 5.2: #10, 18.

Section 5.3: #12.

Problems

Problem 1: (From Problem 11 in Section 4.3) Two ideals I and J of a commutative ring R are called *comaximal* if $I + J = R$.
a. Working in $\mathbb{Z}$, show that $\langle m \rangle$ and $\langle n \rangle$ are comaximal if and only if $\gcd(m, n) = 1$.
b. Let k be an algebraically closed field, and let I and J be ideals of $k[x_1, \ldots, x_n]$. Show that I and J are comaximal if and only if $\mathbf{V}(I) \cap \mathbf{V}(J) = \emptyset$.
c. Show that if I and J are comaximal ideals of a commutative ring R, then $IJ = I \cap J$.

Problem 2: (From Problem 5 in Section 5.3) Let $I = \langle y + x^2 - 1, xy - 2y^2 + 2y \rangle \subseteq \mathbb{R}[x, y]$. As mentioned in the book, the set $G = \{x^2 + y - 1, xy - 2y^2 + 2y, y^3 - (7/4)y^2 + (3/4)y\}$ is a Gröbner basis for I under the $<_{lex}$ ordering with $x > y$.
a. Explain why $\{\overline{1}, \overline{x}, \overline{y}, \overline{y^2}\}$ is a basis for $\mathbb{R}[x, y]/I$ over $\mathbb{R}$. It follows that $\dim_{\mathbb{R}}(\mathbb{R}[x, y]/I) = 4$.
b. Write $\overline{x} \cdot \overline{y^2}$ as an $\mathbb{R}$-linear combination of the basis $\{\overline{1}, \overline{x}, \overline{y}, \overline{y^2}\}$.
c. Determine, with explanation, the set $\mathbf{V}(I)$.
d. Is the ring $\mathbb{R}[x, y]/I$ an integral domain? Explain.