Homework 7: Due Friday, April 30

Exercises

Section 3.4: #8.

Section 4.1: #6, 9, 10.

Section 4.2: #1, 3, 6.

Problems

Problem 1: In class, we discussed how given $f \in k[x]$ for a general field k, we can formally compute f'. I also mentioned that we can detect double roots using this formal derivative. So let k be a field, let $f \in k[x]$, and let $a \in k$. Show that the following are equivalent:

1. There exists $g \in k[x]$ with $f = (x-a)^2 \cdot g$.
2. Both $f(a) = 0$ and $f'(a) = 0$.

Note: Feel free to use any of the usual derivative rules, which can all be verified to hold formally on polynomials over any field.

Problem 2: (From Problem 2 in Section 4.1) Let $J = \langle x^2 + y^2 - 1, y - 1 \rangle \subseteq k[x,y]$. Find, with proof, an example of a polynomial $f \in \mathbf{I}(\mathbf{V}(J))$ such that $f \notin J$.

Problem 3: (From Problem 8 in Section 4.1) Let $W \subseteq k^n$ be a variety. By definition, we can fix $f_1, \dots, f_s \in k[x_1, \dots, x_n]$ with $W = \mathbf{V}(f_1, \dots, f_s)$. If $k = \mathbb{R}$, then using the fact that $a^2 > 0$ for all nonzero $a \in \mathbb{R}$, we also have $W = \mathbf{V}(f_1^2 + \cdots + f_s^2)$. Thus, when $k = \mathbb{R}$, every variety can be written as the set of zeros of *one* polynomial equation. In this problem, we generalize this to all fields k that are *not* algebraically closed.
a. Let $g \in k[x]$ be nonzero, and write $g = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with $a_n \neq 0$. Define the *homogenization* of g to be the polynomial $g^h \in k[x,y]$ given by $g^h = a_n x^n + a_{n-1}x^{n-1}y + \cdots + a_1 x y^{n-1} + a_0 y^n$. Show that g has a root in k if and only if there exists $(a,b) \in k^2 \backslash \{(0,0)\}$ with $g^h(a,b) = 0$.
b. Show that if k is not algebraically closed, then there exists $g_2 \in k[x,y]$ such that $\mathbf{V}(g_2) = \{(0,0)\}$.
c. Show that if k is not algebraically closed, then for all $n \in \mathbb{N}^+$, there exists $g_n \in k[x_1, \dots, x_n]$ such that $\mathbf{V}(g_n) = \{(0,0,\dots,0)\}$. *Hint:* Use induction, together with the special polynomial g_2 from part (b).
d. Show that if k is not algebraically closed, then every variety in k^n can be written as the set of zeros of *one* polynomial equation. *Hint:* If $W = \mathbf{V}(f_1, \dots, f_s) \subseteq k^n$, consider the polynomial $g_s(f_1, \dots, f_s)$.