Homework 6: Due Friday, April 23

Exercises

Section 3.1: #2, 3, 4.

Section 3.2: #5.

Section 3.3: #5, 6, 10.

Problems

Problem 1: (From Problem 4 in Section 3.2). Let k be a field, and let

$$I = \langle x^2 + y^2 + z^2 + 2, 3x^2 + 4y^2 + 4z^2 + 5\rangle \subseteq k[x, y, z].$$

a. Show that $I = \langle x^2 + 3, y^2 + z^2 - 1\rangle$.
b. Show that $\{x^2 + 3, y^2 + z^2 - 1\}$ is the reduced Gröbner basis for I under the $<_{lex}$ ordering.
c. Working over $k = \mathbb{C}$, prove that $\pi_1(\mathbf{V}(I)) = \mathbf{V}(I_1)$.
d. Working over $k = \mathbb{R}$, prove that $\mathbf{V}(I) = \emptyset$ and that $\mathbf{V}(I_1)$ is infinite.
Note: By part (d), the assumption that k is algebraically closed is necessary in the Closure Theorem.

Problem 2: (From Problem 4 in Section 3.3). The tangent surface to the twisted cubic is defined parametrically by

$$\begin{aligned} x &= t + u \\ y &= t^2 + 2tu \\ z &= t^3 + 3t^2u. \end{aligned}$$

At the bottom of p. 135, the book gives a Gröbner basis for the corresponding ideal under a lex order with $t > u > y > y > z$, which shows (via Theorem 3.3.1) that the smallest variety containing the parametric surface is the set of solutions to

$$x^3z - (3/4)x^2y^2 - (3/2)xyz + y^3 + (1/4)z^2 = 0.$$

When working over $\mathbb{C}$, the book then uses the Extension Theorem to argue that every point on this variety appears on the parametric surface, i.e. that for all $(x, y, z) \in \mathbb{C}^3$ satisfying the above equation, there exist $t, u \in \mathbb{C}$ that produce the given (x, y, z). Prove that the same statement is true over $\mathbb{R}$. That is, given $(x, y, z) \in \mathbb{R}^3$ that satisfy the above equation, there exist $t, u \in \mathbb{R}$ that produce the given (x, y, z).
Hint: We can not just apply the Extension Theorem, as $\mathbb{R}$ is not algebraically closed. But we do know that such $t, u \in \mathbb{C}$ exist, and you can use the Gröbner basis at the bottom of p. 135.

Problem 3: (From Problem 7 in Section 3.3). Working over $\mathbb{C}$, let S be the surface parametrized by:

$$\begin{aligned} x &= uv \\ y &= uv^2 \\ z &= u^2. \end{aligned}$$

Using a computer algebra system, a Gröbner basis for the lex order with $u > v > x > y > z$ is:

$$\begin{aligned} g_1 &= u^2 - z \\ g_2 &= uv - x \\ g_3 &= ux - vz \\ g_4 &= uy - x^2 \\ g_5 &= v^2z - x^2 \\ g_6 &= vx - y \\ g_7 &= vyz - x^3 \\ g_8 &= x^4 - y^2z. \end{aligned}$$

Using Theorem 3.3.1, it follows that the smallest variety containing S is the set $W = \mathbf{V}(\{x^4 - y^2z\})$. Working over $\mathbb{C}$, show that W contains a point that is not on S. Moreover, determine all such points, and explain why the Extension Theorem does not apply for them.