Homework 5: Due Tuesday, April 20

Exercises

Section 2.7: #2, 9, 10.

Section 2.8: #1, 2, 3, 8, 11.

Section 3.1: #1.

Problems

Problem 1: Working in $k[x, y]$, let $I = \langle x^2 - y, y^2 - 1\rangle$.
a. Using $<_{lex}$ (with $x > y$), show that $\{x^2 - y, y^2 - 1\}$ is a Gröbner basis for I.
b. Determine, with explanation, whether $x^3y + y^2 - x \in I$.

Interlude: It is straightforward to check that any one-point set in k^n is a variety. Using Lemma 1.2.2, it follows that any finite subset of k^n is a variety. Finding a nice description (such as a Gröbner basis) of the ideal $\mathbf{I}(W)$ for these varieties W is an interesting problem. Recall Problem 2 on Homework 2, where you showed that if $a_1, \dots, a_n \in k$ are distinct, then $\mathbf{I}(\{a_1, \dots, a_n\}) = \langle (x - a_1)\cdots(x - a_n)\rangle$. Since we have a generator of principal ideal in $k[x]$ here, it follows that $\{(x - a_1)\cdots(x - a_n)\}$ is the reduced Gröbner basis of $\mathbf{I}(\{a_1, \dots, a_n\})$.

Let's consider some finite sets in k^2. We start with the case where the first coordinates of the elements of our finite set are distinct. You know that given any two points $(a_1, b_1), (a_2, b_2) \in \mathbb{R}^2$ with $a_1 \neq a_2$, there exists a line with equation $y = mx + c$ that goes through the two points. For three such points, one can always find a parabola. The general statement here is known as *Lagrange Interpolation.* See p. 192-193 of my Algebra notes for more information and motivation.

Problem 2: (From Problem 14 in Section 2.7). Let $(a_1, b_1), \dots, (a_n, b_n) \in k^2$ with $a_1, \dots, a_n$ distinct, and let $W = \{(a_1, b_1), \dots, (a_n, b_n)\}$. Let $h \in k[x]$ be the Lagrange Interpolation polynomial, which is given by

$$h(x) = \sum_{i=1}^{n} b_i \prod_{j \neq i} \frac{x - a_j}{a_i - a_j}.$$

Note that each term in the sum has degree $n-1$, so either $h = 0$ or $\deg(h) \leq n-1$. Also, it is straightforward to check that $h(a_i) = b_i$ for all i (again, see my Algebra notes).
a. Show that h is the only element $g \in k[x]$ with either $g = 0$ or $\deg(g) \leq n - 1$ that satisfies $g(a_i) = b_i$ for all i.
b. Show that $\{(x - a_1)\cdots(x - a_n), y - h(x)\}$ is the reduced Gröbner basis for $\mathbf{I}(W)$, where we use the lexicographic monomial ordering with the variables ordered as $y > x$.
Note: An analogous construction, with the roles of x and y switched, works when the b_i are distinct.

Problem 3: While this problem goes through in a field k with $char(k) \neq 2$, let's work in the field $\mathbb{R}$ for concreteness. Let $W = \{(1, 0), (0, 1), (-1, 0), (0, -1)\} \subseteq \mathbb{R}^2$, and suppose that we want to find the reduced Gröbner basis for $\mathbf{I}(W)$ using the monomial ordering $<_{grlex}$ (with $x > y$). Two natural polynomials that vanish on W are $x^2 + y^2 - 1$ and xy.
a. Show that there exists $f \in \mathbf{I}(W)$ such that $LT(f)$ is not divisible by either x^2 or xy.
b. Find, with proof, the reduced Gröbner basis for $\mathbf{I}(W)$.