Homework 2: Due Friday, April 9

Exercises

Section 1.3: #6, 8.

Section 1.4: #1, 3, 6, 7, 8, 9.

Section 1.5: #11.

Section 2.1: #1.

Problems

Problem 1: Let k be a finite field with q elements. In Problem 1c on Homework 1, you showed that every element of k is a root of the polynomial $x^q - x$. Now show that $\mathbf{I}(k) = \langle x^q - x \rangle$, i.e. that a polynomial in $k[x]$ vanishes on all of k exactly when it is a multiple of $x^q - x$.

Problem 2: Let k be a field.
a. Give a careful proof of the following by induction on $n \in \mathbb{N}^+$: If $a_1, a_2, \ldots, a_n \in k$ are distinct, and $f(x) \in k[x]$ has each a_i as a root, then there exists $g(x) \in k[x]$ with $f(x) = (x - a_1)(x - a_2) \cdots (x - a_n) \cdot g(x)$.
b. Show that if $a_1, a_2, \ldots, a_n \in k$ are distinct, then $\mathbf{I}(\{a_1, a_2, \ldots, a_n\}) = \langle (x - a_1)(x - a_2) \cdots (x - a_n) \rangle$.

Problem 3: Let R be a commutative ring, and let I be an ideal of R. Define $\sqrt{I} = \{a \in R : \text{There exists } m \in \mathbb{N}^+ \text{ with } a^m \in I\}$. The set $\sqrt{I}$ is called the *radical* of I.
a. Show that $\sqrt{I}$ is an ideal of R.
b. Show that if P is a prime ideal of R with $I \subseteq P$, then $\sqrt{I} \subseteq P$.
Aside: Part (b) implies that $\sqrt{I}$ is contained in the intersection of all prime ideals that contain I. In fact, $\sqrt{I}$ is *equal* to the intersection of all prime ideals that contain I, but the other containment is significantly harder.