Homework 8: Due Wednesday, April 12

Problem 1: Let G be a group, and let $Aut(G)$ be the set of all automorphisms of G. By Proposition 6.3.3, we know that the composition of two automorphisms is an automorphism, and the inverse of an automorphism is an automorphism. Thus, $(Aut(G), \circ, id_G)$ is a group. For each $g \in G$, we know from Problem 4 on Homework 7 that the function $\varphi_g \colon G \to G$ defined by $\varphi_g(a) = gag^{-1}$ is an automorphism, so $\varphi_g \in Aut(G)$. Define $\psi \colon G \to Aut(G)$ by letting $\psi(g) = \varphi_g$. Show that ψ is a homomorphism.

Problem 2: Compute, with explanation, the conjugacy classes and the Class Equation for D_5.
Note: Use ideas from class and results about both conjugation and D_n from past homework to cut down on computational work. In fact, it is possible to compute the Class Equation first, and then use it and old homework to determine most of the conjugacy classes.

Problem 3: Determine (with proof) which finite groups have exactly two conjugacy classes.

Problem 4: Suppose that G is a nonabelian group with $|G| = 125$. Show that $|Z(G)| = 5$ and that $G/Z(G) \cong \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$.

Problem 5: Let $n, p \in \mathbb{N}^+$ and assume that p is prime. Let

$$X = \{1, 2, \dots, n\}^p = \{(a_1, a_2, \dots, a_{p-1}, a_p) : 1 \leq a_i \leq n \text{ for all } i\}$$

be the set of all p-tuples such that each coordinate is an integer between 1 and n. Notice that S_p acts on X via

$$\sigma * (a_1, a_2, \dots, a_{p-1}, a_p) = (a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(p-1)}, a_{\sigma(p)}).$$

Let $H = \langle (1\ 2\ \dots\ p) \rangle \subseteq S_p$, so $|H| = p$. Since H is a subgroup of S_p, we know that H acts on X via the above action as well. For example,

$$(1\ 2\ \dots\ p) * (a_1, a_2, \dots, a_{p-1}, a_p) = (a_2, a_3, \dots, a_p, a_1),$$

so $(1\ 2\ \dots\ p)$ cyclically shifts an element in X to the left by 1. Similarly, $(1\ 2\ \dots\ p)^2$ cyclically shifts to the left by 2, etc.
a. Show that every orbit of the action of H on X has size either 1 or p.
b. Show that there are exactly n orbits of size 1.
c. Show that $p \mid (n^p - n)$.
Note: This gives another proof of Fermat's Little Theorem.

Problem 6: This problem provides another proof of Cauchy's Theorem (so don't use Cauchy's Theorem at any point!). Let G be a group and suppose that p is a prime that divides $|G|$. Let

$$X = \{(a_1, a_2, \dots, a_{p-1}, a_p) \in G^p : a_1 a_2 \cdots a_{p-1} a_p = e\},$$

i.e. X consists of all p-tuples of elements of G such that when you multiply them in the given order, you obtain the identity. For example, if $G = S_3$ and $p = 3$, then $((1\ 2), id, (1\ 2)) \in X$ and $((1\ 2), (1\ 3), (1\ 2\ 3)) \in X$, but $(id, (1\ 2), (1\ 2\ 3)) \notin X$.
a. Show that $|X| = |G|^{p-1}$.
b. Show that if $(a_1, a_2, \dots, a_{p-1}, a_p) \in X$, then $(a_2, a_3, \dots, a_p, a_1) \in X$.

Interlude: Let $H = \langle (1\ 2\ \dots\ p) \rangle \subseteq S_p$, so $|H| = p$. Part (b) says that any cyclic shift of an element of X is also in X, so H acts on X as in Problem 5.

c. Notice that $|\mathcal{O}_{(e,e,\dots,e,e)}| = 1$. Show that there exists at least one other orbit of size 1.
d. Conclude that G has an element of order p.