

Homework 6: Due Wednesday, March 15

Problem 1: Suppose that H and K are both normal subgroups of G . Show that $H \cap K$ is a normal subgroup of G .

Problem 2: Let G be a group and let H be a normal subgroup of G .

a. Show that G/H is abelian if and only if $a^{-1}b^{-1}ab \in H$ for all $a, b \in G$.

b. Suppose $[G : H]$ is finite and let $m = [G : H]$. Show that $a^m \in H$ for all $a \in G$.

Problem 3: Let G and H be groups. Show that $G \times H \cong H \times G$.

Problem 4: Determine, with proof, whether the following pairs of groups are isomorphic.

a. A_6 and S_5 .

b. $\mathbb{Z}/84\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/14\mathbb{Z}$.

c. $U(\mathbb{Z}/18\mathbb{Z})$ and $\mathbb{Z}/6\mathbb{Z}$.

d. S_4 and $\mathbb{Z}/6\mathbb{Z} \times U(\mathbb{Z}/5\mathbb{Z})$.

e. A_4 and D_6 .

f. $U(\mathbb{Z}/5\mathbb{Z})$ and $U(\mathbb{Z}/10\mathbb{Z})$.

g. $S_3 \times \mathbb{Z}/2\mathbb{Z}$ and A_4 .

h. $D_4/Z(D_4)$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Problem 5: Consider the group $G = \mathbb{R} \setminus \{-1\}$ with operation $a * b = a + b + ab$ from Homework 3. Let H be the group $\mathbb{R} \setminus \{0\}$ with operation equal to the usual multiplication. Show that $G \cong H$.

Problem 6: Consider the group $G = U(\mathbb{Z}/15\mathbb{Z})$. Find nontrivial cyclic subgroups H and K of G such that G is the internal direct product of H and K . Use this to find $m, n \in \mathbb{N}$ with $m, n \geq 2$ such that $U(\mathbb{Z}/15\mathbb{Z}) \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$.